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The objectives of this project were (1) to select those problems faced continually by department chairmen, deans and college presidents, which relate to the efficiency with which educational resources are used, (2) to state essential features of these problems in mathematical form and identify their mathematical structures with management science models used in other fields, and (3) to apply the appropriate solution methods to synthetic models of academic departments, divisions, or small colleges. The approach taken includes (1) a logical formulation, (2) a mathematical formulation based on management science concepts, (3) the construction of a non-trivial example based on academi c organization and staffing patterns, teaching loads, salary levels, and (4) the completion of one or more sequences of calculations showing how management science techniques would improve the results of resource allocation or decision-making process over specified conventional or traditional procedures. The report emphasizes three models which should be useful aids to resource allocation and other major decision processes in educational institutions. (BC)

FINAL REPORT

Contract No. OEC-3-6-068058

FORMULATION OF MANAGEMENT SCIENCE MODELS FOR SELECTED PROBLEMS OF COLLEGE ADMINISTRATION

by:

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HEALTH, EDUCATION AND WELFARE

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CONTENTS

Page

LIST OF TABLES

Table		Page

FORMULATION OF MANAGEMENT SCIENCE MODELS FOR SELECTED PROBLEMS OF COLLEGE ADMINISTRATION

by:

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During the past few years there has been a rapid development of the field called management science (not to be confused with the older tradition of "scientific management" in the sense of time and motion studies, etc.). Management science has made major contributions to the solution of problems involving the optimal allocation of limited resources, decision-making under conditions of uncertainty, and decision processes involving two or more stages (as in an administrative hierarchy). The central concepts involved can be stated in concise mathematical terms and can be extended to a very wide range of applied fields in addition to those in which successes have already been scored (business management, military operations research, traffic engineering, control systems engineering, etc.).

College administrators must deal with problems of allocating limited resources and must continually make decisions in the face of uncertainty as to enrollments, losses of faculty, future budget limitations, developments in secondary schools and changing job opportunities for college graduates. The specific objectives of this project are:

1. To select a number of problems that are faced continuously by department chairmen, deans and college presidents and are of great importance to the efficiency with which our educational resources are used;

2. To state the essential features of these problems in mathematical form and identify their mathematical structures with management science models used in other fields; and

3. To apply the appropriate solution methods to synthetic or illustrative models of academic departments, divisions, or small colleges.

I. The Problems Considered

A. General Statement

After twelve years as head of a large academic department and after many, many conversations with faculty members, chairmen and deans in various colleges and universities, it appears to the principal investigator that academic administration is making little use of modern scientific approaches to resource allocation and optimal decision processes. 1/ Few problems are specified with sufficient clarity and rigor, and with sufficient attention to the definition of the relevant variables, that two objective observers could take the problem as described and arrive at the same quantitative decision.

This is not to say that all aspects of academic life are or should be subject to quantification. A good college faculty includes proud, sensitive and creative people. However, faculties, and their chairmen,

1/ This is something quite different from automation and computerization of recording, accounting and reporting activities, as will be seen shortly.

deans and presidents, must make many decisions which affect in quantitative ways the "inputs" and "outputs" of the educational processes in which they are engaged.

Almost without exception, college presidents are keenly aware of pressures from deans and faculty members to do more and better things--things which cannot be done with the resources available. To an economist, this involves the classical economic problem of allocating limited resources among competing uses.

Economists often view decision makers as using price systems to aid in the management of their enterprises. These concepts are seldom explicitly used by educational administrators. The inputs of educational processes are measured and priced although few colleges, to the best of our knowledge, measure or price them in ways that aid educational decision-making. The measurement and pricing procedures used are more commonly designed only to insure that the college stays within its budget for the year. Recent developments in management science have emphasized the concept of a "shadow price" which does not necessarily equal the market price of a resource but which shows the amount by which the value of output of an enterprise could be increased if one additional unit of a particular scarce resource were made available.

Some of the "outputs" of colleges are not explicitly priced by college faculty members and administrators. Some of them do have value, however, in terms of the greater earning power and leadership capabilities of college graduates. In recent years, T. W. Schultz (18), Gary Becker (2), and others have made progress in quantifying the vocational value as contribution to career income resulting from a college education--this is, in a sense, the value of one educational output to the student who receives the education.

The value of a particular output may be estimated at different levels by different faculty members or administrators. However, the assignment of illustrative prices to the various outputs at least challenges others to state their own judgment as to relative prices and their reasons for them. It may turn out that some decisions are not sensitive to the differences between alternative sets of prices which two different outside observers as expert panels would regard as reasonable.

This project has not been concerned with improving estimates of prices of educational inputs and outputs. We have drawn on other research both here and elsewhere for estimated or illustrative prices. We are concerned with the procedures for making reproducible decisions which are in some sense optimal <u>given</u> the estimated or measured prices of inputs and outputs and quantitatively specified budget personnel and space constraints. Knowledge that procedures exist for making good or even optimal use of such data may do more than anything else to stimulate improvements in the accuracy and relevance of the data collected in the future.

No one knows by how much the efficiency of college education in the United States could be increased. Given a set of institutions which have prided themselves on the intangibility of their product, research focused upon efficient resource allocation and improved decision processes with respect to quantitative aspects of college administration should have an ultimate payoff running into many hundreds of millions of dollars. This is not, of course, the patented payoff of any one project but the cumulative payoff of an "optimizing" approach and attitude by faculty members and administrators becoming gradually more prevalent over a period of years.

B. References to Related Work

Some progress has been made in the general area of the economics of education and a good deal of progress has been made in the tools needed for making optimal decisions. Becker (2), T. W. Schultz (18) and others have attempted to estimate the value to the holders and to society of various types of education. Adelman (1), Bowles (4), Stone (19), and others have formulated planning models of several national educational systems. Dantzig and Wolfe (8) have developed an algorithm which under some circumstances can be used to compute optimum solutions to decision-making problems of large organizations. Kornai and Liptak (13) have dealt with the same problem and have shown the analogy between decentralized decision-making and certain matrix games. Day (9) has made extensive use of a recursive programming model in which the levels at which activities can be carried on in Year t are dependent upon the levels attained in the preceding year.

Iowa State University economists have also been active in this area. Winkelmann (20) has constructed a model for allocating faculty members among various teaching and research assignments. Plessner, Fox and Sanyal (17) have constructed for an individual department a dynamic programming policy model which determines, given the size and characteristics of the initial faculty, initial enrollments, and input and output prices, the optimal admissions andoutput pattern for a four-year planning period. McCamley (15) has shown that the results of Kornai and Liptak (13) and Dantzig and Wolfe (8) may be combined to provide the basis for decision-making procedures that could be adopted by educational institutions. Fox and Sengupta (10) have reviewed much of the extant literature dealing with educational planning and have indicated some of the features that should be included in models of educational departments.

C. Specific Models Treated in This Report

For each of a number of major types of problems of college administration our approach includes (1) a logical formulation, (2) a mathematical formulation based on management science concepts, (3) the construction of a non-trivial example, based on academic organization and staffing patterns, teaching loads, salary levels, and the like which are within the range of current academic experience in the United States and (4) the completion of one or more sequences of calculations showing how management science techniques would improve the results of resource allocation or decision-making process over specified conventional or traditional procedures.

In our project proposal we listed "some of the problems that will most likely be conceptualized" as:

a. The use of a linear programming model to allocate a stipulated faculty among courses (and between courses, research, and administrative activities) for a single year, given projected enrollments and desire for courses on the part of students;

b. The use of a recursive programming model to follow successive decisions of a department chairman or faculty over a period of years;

c. The modeling of a two-level decision-making process;

d. Estimating the relative contributions of different decision makers to an allocation procedure; and

e. The use of stochastic programming to estimate the value of certain types of information.

As it turned out we went quite deeply into some problems and less deeply or not at all into others. The total amount of effort put into the project and directly-related research far exceeded the amount provided by the project contract ($7,500). 2/

In this report we will emphasize three approaches or models which should prove to be useful aids to resource allocation and other major decision processes in educational institutions.

2/ The principal disbursements from the contract funds were salaries for Francis McCamley (6 months) and Yakir Plessner (3 months). Plessner, Fox, Von Hohenbalken and Sengupta did some related research on Iowa State University funds, and McCamley did related research for nine additional months while supported on an NDEA fellowship.

II. Models for Allocating Given Staff Resources Among Fixed Teaching and Research Comitments in a Single Year

The first problem to be discussed is that of finding the best allocation of given staff resources among fixed teaching and research commitments. This problem could arise in situations in which the college dean or some other official has specified the number of faculty members which a department may employ and in which enrollment and other considerations have specified the courses (and numbers of sections) that must be taught and the research projects that must be completed.

The objectives or goals of the department chairmen help define what the best allocation is. For example, Winkelmann (20) has suggested that department chairmen might attempt to maximize their departments' contribution to national income. Hamelman (11) suggests minimizing the proportion of students failing--i.e., failing despite conscientious effort.

A. The Basic Model

In one of its simplest forms the problem of allocating staff members among alternative tasks can lead to a model which is formally identical with the transportation model. As a result there exist many ways of obtaining numerical solutions to the problem. Such a formulation also has the advantage that the optimum solutions can easily be restricted to integers.[3/]

3/ If both the numbers of units of resources available and the numbers of units required for each course are integers, one of the optimal solutions will always consist only of integers.

In order to use the transportation type of model certain requirements must be met by the problem.

First, both the inputs (faculty time) and the input requirements per course or research project must be measured in the same units. This is a trivial requirement in most cases and can probably best be fulfilled by adopting as the unit of measurement the amount of time required to teach one section of one course. Teaching load per faculty member is often measured in this manner anyway. Research inputs are sometimes also measured in this manner too, especially when the time devoted to research is measured in terms of the reduction of teaching loads from that required by some full-time teaching load norm.

Second, the total staff resources available must equal the amount required to meet all of the departments' teaching and research commitments. If the staff resources exceed the amount needed to meet all commitments this second requirement may be met by adding artificial commitments (i.e., units of free time) to take up the slack. If the staff resources are less than the amount required, there is no feasible allocation and either the commitments or the number of staff members must be adjusted.

Third, any staff member must be capable of teaching any section of any course that must be taught, or of completing any portion of any research project. This does not exclude the possibility that some staff members might do quite badly in some assignments. (If the effectiveness of faculty member i in course j is judged to be extremely low, this assignment will rarely appear in the optimal solution.)

Fourth, the contribution which a particular staff member makes if he teaches a particular course (or conducts a particular research project) must be a homogeneous linear function only of the number of units of time he devotes to that course (or research project). It must not depend upon the amount of his time allocated to other courses (or projects) nor upon the allocations of other staff members' time. In other words, if the value of one section of course j taught by faculty member i is rated at 10 units, the total value of two sections of course j taught by faculty member i is rated at 20 units, of three sections at 30 units, and so on.[4/]

The mathematical model is presented in the appendix. This model requires three types of information.

The first type of information required is a list of faculty members and the amount of teaching and research inputs available from each of them.

The second type of information required is a list of courses which must be taught (and research projects which must be completed) and the number of units of input required for each.[5/]

4/ More complicated models could be devised to take account of favorable effects of variety. Or, we could impose upper limits on the number of units of a faculty member's time that could be assigned to any one course. But in the models described in this report, we assume that all of a faculty member's time could be assigned to a single course without diminution in his "value per section."

5/ The number of units of inputs required per course usually equals the number of sections to be taught.

The third type of information required is a set of objective function weights. These weights should indicate for the ith staff member and for the jth task the contribution that would be made to the departments' objectives if the ith staff member supplied one unit of input for the jth task. In total, n · m of these weights are required, where n is the number of faculty members and m is the number of different tasks.

The information required for this model could be presented in the following form:

$$
\begin{array}{ccccc}
c_{11} & c_{12} & \cdots\cdots\cdots & c_{1m} & a_1 \\
c_{21} & c_{22} & \cdots\cdots\cdots & c_{2m} & a_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
c_{n1} & c_{n2} & \cdots\cdots\cdots & c_{nm} & a_n \\
b_1 & b_2 & & b_m &
\end{array}
$$

The a_i's indicate the amounts of inputs available from the various faculty members. The b_j's indicate the amount of inputs required by each of the various tasks. The c_{ij}'s are the objective function weights.[6/]

6/ In the mathematical programming literature, "objective function" is a technical term denoting a combination of activity levels and weights which is to be maximized. "Objective" is used in the sense of "goal" or "target." The weights included in the objective function might be market prices in some applications; in the faculty assignment problem, they would more likely represent the judgments of a department chairman.

For small problems such a model may be solved by hand. The solution procedure consists of two stages. During the first stage a feasible solution is obtained. During the second stage an optimal solution is obtained.

The first stage consists of a single step which is repeated until feasibility is obtained. Choose the largest of the c_{ij}'s for which both b_j and a_i are greater than zero. If this element is c_{rs}, set x_{rs} equal to the smaller of the current values of b_r and a_s. Update the values of b_r and a_s by subtracting x_{rs} from both. This step is repeated until all b_j's and a_i's are equal to zero.

During the second stage the solution is improved until an optimum solution is obtained. The only way to improve the solution is to increase some allocation vector (one of the x_{ij}'s) from a zero level to a positive level. The first step involves determining which activity level to increase. Usually most of the activity levels will be zero and in addition there will often be several ways of increasing any given activity level. In such a case the easiest way to determine which activity level to increase is to first solve the dual of the model.7/ The dual variables u_i (the marginal value of a unit of input supplied by the ith staff member) and v_j (the marginal value of a unit of input demanded by the jth task) must satisfy the relationship

7/ For a discussion of the meaning of the "dual" of a linear programming model, see Dorfman, R., P. Samuelson and R. Solow, Linear Programming and Economic Analysis (McGraw-Hill: New York), 1958, pp. 100-104 and 122-127.

$$c_{ij} = u_i - v_j$$

if x_{ij} is greater than zero. This leads to a system of n + m - 1 (or fewer) equations which can easily be solved for the u_i's and v_j's. Those x_{ij}'s for which $c_{ij} - u_i + v_j$ is greater than zero are candidates for increases in activity levels. The x_{ij} to increase first is the one for which $c_{ij} - u_i + v_j$ is the largest.

The next step involves determining how to increase the activity level. In order to increase the level of any activity (say x_{rs}) it is necessary to decrease the levels of at least two other activities and increase the level of at least one other activity. The method of changing the level of activity x_{rs} should be chosen so that

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \Delta x_{ij} c_{ij}$$

is maximized subject to

$$\sum_{i=1}^{n} \Delta x_{ij} = 0 \qquad j = 1, 2, \ldots, m$$

$$\sum_{j=1}^{m} \Delta x_{ij} = 0 \qquad i = 1, 2, \ldots, n$$

and

$$\Delta x_{rs} = 1, \quad x_{ij} + \Delta x_{ij} \geq 0 \qquad \begin{array}{l} i = 1, 2, \ldots, n \\ j = 1, 2, \ldots, m \end{array}$$

Once the best way of changing x_{rs} is determined, x_{rs} is set equal to the largest value permitted by that method of changing the activity level.

The second stage steps are repeated until at some point

$$c_{ij} \leqq u_i - v_j \qquad \text{for} \quad i = 1, 2, \ldots, n$$
$$j = 1, 2, \ldots, m .$$

At that point an optimum solution has been obtained.

A couple of examples may serve to clarify some of the ideas discussed above.

Consider an extremely small department which has two faculty members each of whom supplies enough inputs for three sections per quarter; the department offers only two distinct courses. Four sections of one course and two sections of the other must be taught in a particular quarter. The information needed for the model can be summarized as follows:

	Course 1 Economic principles	Course 2 Theory of the firm	
Faculty Member 1	$c_{11} = 10$	$c_{12} = 7$	$a_1 = 3$
Faculty Member 2	$c_{21} = 6$	$c_{22} = 8$	$a_2 = 3$
	$b_1 = 4$	$b_2 = 2$	

Part of this information can be expressed as a set of equations, as follows:

$$
\begin{array}{lllll}
x_{11} + x_{12} & & & & = 3 \\
& & x_{21} + x_{22} & & = 3 \\
-x_{11} & & -x_{21} & & = -4 \\
& -x_{12} & & -x_{22} & = -2
\end{array}
$$

The objective function to be maximized is

$$W = 10x_{11} + 7x_{12} + 6x_{21} + 8x_{22} \quad .$$

The first four equations can also be written in matrix equation form as:

$$
\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix}
\begin{bmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \end{bmatrix}
=
\begin{bmatrix} 3 \\ 3 \\ -4 \\ -2 \end{bmatrix}
$$

The complete set of information can be displayed as in Table 1. A unit of activity x_{11} assigns one unit of the time of Faculty Member 1 to Course 1; a unit of activity x_{12} assigns one unit of Faculty Member 1's time to Course 2; a unit of activity x_{21} assigns one unit of Faculty Member 2's time to Course 1; and a unit of activity x_{22} assigns one unit of Faculty Member 2's time to Course 2.

Table 1. Faculty Allocation Model: Basic Information

	Activities:				Availabilities and requirements
Activity levels (x_{ij}):	x_{11}	x_{12}	x_{21}	x_{22}	

Possible assignment activities:

$$\begin{bmatrix} 1 & 1 & & \\ & & 1 & 1 \\ -1 & & -1 & \\ & -1 & & -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -4 \\ -2 \end{bmatrix}$$

Objective function weights (c_{ij}):

$$\begin{bmatrix} 10 & 7 & 6 & 8 \end{bmatrix} \quad \text{maximum}$$

Initially limiting factors:

Availabilities: a_i

$$\left[\begin{array}{cc:cc} 3 & 3 & 3 & 3 \end{array}\right]$$

Requirements: b_j

$$\left[\begin{array}{cc:cc} -4 & -2 & -4 & -2 \end{array}\right]$$

To obtain a feasible solution which will achieve a high value (but not necessarily the highest value) of the objective function W, we introduce the activities (and establish their levels) in the following order (Table 2).

At the end of the first step

$$x_{11} = 3, \quad x_{12} = 0, \quad x_{21} = 0, \quad x_{22} = 0,$$

$$a_1 = 0, \quad a_2 = 3, \quad b_1 = 1, \quad b_2 = 2.$$

At the end of the second step

$$x_{11} = 3, \quad x_{12} = 0, \quad x_{21} = 0, \quad x_{22} = 2,$$

$$a_1 = 0, \quad a_2 = 1, \quad b_1 = 1, \quad b_2 = 0.$$

At the end of the third step

$$x_{11} = 3, \quad x_{12} = 0, \quad x_{21} = 1, \quad x_{22} = 2,$$

$$a_1 = a_2 = b_1 = b_2 = 0.$$

All commitments are met and all of the available time is allocated, so Step 3 gives us our first feasible solution.

This solution can be written in matrix form as

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -4 \\ -2 \end{bmatrix}$$

Table 2. Faculty Allocation Model: Steps Toward First Feasible Solution

	Step Number:			
	0	1	2	3
		Activity Introduced:		
		x_{11}	x_{22}	x_{21}
		Units Assigned:		
		3	2	1
a_1	3	0	0	0
a_2	3	3	1	0
$-b_1$	-4	-1	-1	0
$-b_2$	-2	-2	0	0
c_{ij}		10	8	6
$c_{ij}\ x_{ij}$		30	16	6
$\sum_{i=1}^{2} \sum_{j=1}^{2} c_{ij}\ x_{ij}$		30	46	52

Because $(a_1 + a_2) = (b_1 + b_2)$, whenever we find a set of activities which satisfies three of these restrictions exactly that same set will also satisfy the fourth. We can therefore drop the fourth equation (row) from the above set, and eliminate the column corresponding to activity x_{12}, which is not used in the feasible solution. We then have simply

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}, \text{ or } Ax = r.$$

To compute the dual of this last matrix equation, we write down a new A matrix (call it B) in which Column 1 is equal to Row 1 of the A matrix, Column 2 to Row 2, and Column 3 to Row 3.[8/] We then write

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ v_1 \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 6 \end{bmatrix}, \quad Bu = c.$$

This is the dual we require to test whether the first feasible solution is also optimal.

We can solve for u by using the inverse matrix, B^{-1}, as $u = B^{-1}c$. The inverse matrix turns out to be

---

8/ See Dorfman, Samuelson and Solow, op. cit., p. 101.

$$B^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

Hence,

$$\begin{bmatrix} u_1 \\ u_2 \\ v_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 10 \\ 8 \\ 6 \end{bmatrix}$$

from which

$$\begin{aligned} u_1 &= 10 + 8 - 6 = 12 ; \\ u_2 &= 0 + 8 + 0 = 8 ; \\ v_1 &= 0 + 8 - 6 = 2 ; \\ v_2 &= 0. \end{aligned}$$

Then

$$\begin{aligned} c_{11} &\leq u_1 - v_1, \quad \text{or} \quad 10 \leq 12 - 2 ; \\ c_{12} &\leq u_1 - v_2, \quad \text{or} \quad 7 \leq 12 - 0 ; \\ c_{21} &\leq u_2 - v_1, \quad \text{or} \quad 6 \leq 8 - 2 ; \end{aligned}$$

and $c_{22} \leq u_2 - v_2, \quad \text{or} \quad 8 \leq 8 - 0 .$

We can/ not increase $W = 10x_{11} + 7x_{12} + 6x_{21} + 8x_{22} = 52$ by changing the level of any x_{ij}, so the first feasible solution is also the optimal solution.

It may be noted in passing that the dual also has an objective function, which is to <u>minimize</u>

$$D = u_1a_1 + u_2a_2 + v_1b_1 + v_2b_2 .$$

The minimum value of D is equal to the maximum value of W. Thus, in the present case,

$$D_{min} = 12(3) + 8(3) + 2(-4) + 0(-2) , \quad \text{or}$$

$$D_{min} = 36 + 24 - 8 + 0 = 52 = W_{max} .$$

In the present small example, of course, we do not need all the above paraphernalia to determine that the first solution is also optimal. For what would happen if we increased activity x_{12} by one unit, from 0 to 1?

1. The added unit of x_{12} is worth seven points ($c_{12} = 7$).

2. To get it, we must withdraw one unit of x_{11} worth ten points ($c_{11} = 10$), leaving one section of Course 1 unassigned.

3. However, for the moment we have assigned three units of faculty time to Course 2, for which only two units are needed. Thus, we withdraw a unit of Faculty Member 2's time from Course 2, where it is worth eight points, and assign it to Course 1, where it is worth only six points.

Thus, in order to increase the level of x_{12} by one unit we have had to subtract one unit from each of two other activities (x_{11} and x_{22})

and add one unit to a fourth activity (x_{21}). After all this rearranging, we find that

$$W = 10(2) + 7(1) + 6(2) + 8(1), \quad \text{or}$$

$$W = 20 + 7 + 12 + 8 = 47 .$$

The objective function has been altered in the amount

$$(c_{12}-c_{11}) + (c_{21}-c_{22}) = (7-10) + (6-8) = -3 + (-2) = -5 .$$

When the numbers of faculty members and courses are considerably larger than two and two, it is convenient to use standard computerized methods for determining the optimal solution and the shadow prices associated with it. Iterative calculations, matrix inversions and the like are carried on in the computer and the final solution is printed out, along with other measures such as the shadow prices, which aid in interpreting the solution as such.

A slightly larger example may serve to illustrate the second stage computations. Consider a department which has four faculty members each of whom teaches nine sections per year or contributes equivalent inputs to teaching and research assignments. It also offers six courses which during a given year it must offer at the rate of 9, 7, 6, 5, 4, and 3 sections per year. It also is conducting two small research projects which require respectively two units of faculty inputs and one unit of faculty input. It will be assumed that scheduling of sections among quarters and hours of the day is sufficiently flexible that the allocation for a whole year can be obtained without worrying about which quarter a particular section will be taught.

The information relevant to this problem is given in compact form in Table 3. Table 4 spells out in detail the 32 possible activities. One unit of activity x_{ij} assigns one unit of the time of Faculty Member i to Course j. This subtracts one unit from a_i and adds one unit to b_j. The a_i's are given positive signs; they are stocks or surpluses to be drawn down ultimately to zero. The b_j's are given negative signs; they are needs or deficits to be satisfied or made good until ultimately no deficits remain. In Table 4, all the elements in the 12 by 32 matrix which are not either 1 or -1 are zero.

Starting from Table 4, we can arrive at a first feasible solution as before, assigning the time of each faculty member i to tasks j for which his c_{ij}'s are relatively high. The steps are shown in Table 5. We have not been meticulous about bringing the very highest c_{ij}'s in first, but have come fairly close to this. For example, $b_5 = 4$, so only four units of Faculty Member 1's time can be assigned to activity x_{15}, which has the highest c_{ij} of all ($c_{15} = 15$); we assign the other five units of Faculty Member 1's time to activity 11, which is the second most productive use of his time ($c_{11} = 10$). Activities x_{27} and x_{22} are the two best uses of Faculty Member 2 ($c_{27} = 12$ and $c_{22} = 8$). We assign five units of Faculty Member 3's time to activity x_{34}, which is

Table 3. Faculty Allocation Problem With Four Faculty Members, Six Courses, and Two Research Projects

Faculty Member	Courses						Research		Time Units Available
	1	2	3	4	5	6	7	8	a_i
1	10	7	5	9	15	4	6	3	9
2	6	8	4	3	7	5	12	2	9
3	7	7	6	11	10	9	5	1	9
4	5	6	2	5	3	8	7	6	9
Time Units Required: b_j	9	7	5	5	4	3	2	1	36

Table 4. Faculty Allocation Problem: Detailed Statement of all 32 Possible Activities, the 32 Weights, and the 12 Availability and Requirement Restrictions

Assignments (x_{ij}):		x_{11}	x_{12}	x_{13}	x_{14}	x_{15}	x_{16}	x_{17}	x_{18}	:	x_{21}	x_{22}	x_{23}	x_{24}	x_{25}	x_{26}	x_{27}	x_{28}	:	x_{31}
Faculty Members	1	1	1	1	1	1	1	1	1											
	2										1	1	1	1	1	1	1	1		
	3																			1
	4																			
Courses and Research Projects	1	-1									-1									-1
	2		-1									-1								
	3			-1									-1							
	4				-1									-1						
	5					-1									-1					
	6						-1									-1				
	7							-1									-1			
	8								-1									-1		
Objective Function Weights (c_{ij})′		10	7	5	9	15	4	6	3	:	6	8	4	3	7	5	12	2	:	7
Initially Limiting Factors:																				
Availabilities (a_i)		9	9	9	9	9	9	9	9	:	9	9	9	9	9	9	9	9	:	9
Requirements (b_j)		-9	-7	-5	-5	-4	-3	-2	-1	:	-9	-7	-5	-5	-4	-3	-2	-1	:	-9

Table 4. (Cont.)

Assignments (x_{ij}):		x_{32}	x_{33}	x_{34}	x_{35}	x_{36}	x_{37}	x_{38}	x_{41}	x_{42}	x_{43}	x_{44}	x_{45}	x_{46}	x_{47}	x_{48}		Availabilities and Requirements
Faculty Member	1																	9
	2																	9
	3	1	1	1	1	1	1	1										9
	4								1	1	1	1	1	1	1	1		9
Courses and Research Projects	1								-1									-9
	2	-1								-1								-7
	3		-1								-1							-5
	4			-1								-1						-5
	5				-1								-1				=	-4
	6					-1								-1				-3
	7						-1								-1			-2
	8							-1								-1		-1
Objective Function Weights (c_{ij})		7	6	11	10	9	5	1	5	6	2	5	3	8	7	6		
Initially Limiting Factors:																		
Availabilities (a_i)		9	9	9	9	9	9	9	9	9	9	9	9	9	9	9		
Requirements (b_j)		-7	-5	-5	-4	-3	-2	-1	-9	-7	-5	-5	-4	-3	-2	-1		

Table 5. Faculty Allocation Problem: Steps Toward First Feasible Solution

Faculty Member (F_i) and Courses and Projects (R_j)	Step Number: 0	1	2	3	4	5	6	7	8	9	10
Activity Introduced:		x_{11}	x_{15}	x_{22}	x_{27}	x_{31}	x_{34}	x_{36}	x_{41}	x_{43}	x_{48}
Units Assigned:		5	4	7	2	1	5	3	3	5	1
F_1	9	4	0	0	0	0	0	0	0	0	0
F_2	9	9	9	2	0	0	0	0	0	0	0
F_3	9	9	9	9	9	8	3	0	0	0	0
F_4	9	9	9	9	9	9	9	9	6	1	0
R_1	-9	-4	-4	-4	-4	-3	-3	-3	0	0	0
R_2	-7	-7	-7	0	0	0	0	0	0	0	0
R_3	-5	-5	-5	-5	-5	-5	-5	-5	-5	0	0
R_4	-5	-5	-5	-5	-5	-5	0	0	0	0	0
R_5	-4	-4	0	0	0	0	0	0	0	0	0
R_6	-3	-3	-3	-3	-3	-3	-3	0	0	0	0
R_7	-2	-2	-2	-2	0	0	0	0	0	0	0
R_8	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0
c_{ij}		10	15	8	12	7	11	9	5	2	6
$x_{ij}\ c_{ij}$		50	60	56	24	7	55	27	15	10	6
$\Sigma x_{ij}\ c_{ij}$		50	110	166	190	197	252	279	294	304	310

his best use; however, his second most productive use, in Course 5 ($c_{35} = 10$), is no longer available as all four sections of Course 5 have been allocated to Faculty Member 1. We therefore assign the last four units of Faculty Member 3's time to activities x_{36} and x_{31}, in which his productivity is rated at 9 and 7 points respectively. Faculty Member 4, on this first round, is assigned to whatever tasks are left over, namely to activities x_{41}, x_{43} and x_{48} in which his productivity is rated at 5, 2 and 6 points respectively.

Table 6 shows the first feasible solution in matrix equation form. The solution is given by $x_{11} = 5$, $x_{15} = 4$, $x_{22} = 7$, $x_{27} = 2$, $x_{31} = 1$, $x_{34} = 5$, $x_{36} = 3$, $x_{41} = 3$, $x_{43} = 5$, $x_{48} = 1$. The corresponding dual solution (from Table 7) is given by $u_1 = 15$, $u_2 = 12$, $u_3 = 12$, $u_4 = 10$, $v_1 = 5$, $v_2 = 4$, $v_3 = 8$, $v_4 = 1$, $v_5 = 0$, $v_6 = 3$, $v_7 = 0$ and $v_8 = 4$.

To obtain these values of the u_i's and v_j's, we note that each of the ten rows of the matrix equation in Table 7 is itself an ordinary algebraic equation:

Row 1: $u_1 - v_1 = 10$

Row 2: $u_1 - v_5 = 15$

Row 3: $u_2 - v_2 = 8$

Row 4: $u_2 - v_7 = 12$

Row 5: $u_3 - v_1 = 7$

Row 6: $u_3 - v_4 = 11$

Table 6. Faculty Allocation Problem: First Feasible Solution in Matrix Equation Form

Faculty Member (F_i) and Courses and Projects (R_j)	Activities in First Feasible Solutions										Units Assigned:		Availabilities and Requirements
	x_{11}	x_{15}	x_{22}	x_{27}	x_{31}	x_{34}	x_{36}	x_{41}	x_{43}	x_{48}			
F_1	1	1	0	0	0	0	0	0	0	0	5		9
F_2	0	0	1	1	0	0	0	0	0	0	4		9
F_3	0	0	0	0	1	1	1	0	0	0	7		9
F_4	0	0	0	0	0	0	0	1	1	1	2		9
R_1	-1	0	0	0	-1	0	0	-1	0	0	1		-9
R_2	0	0	-1	0	0	0	0	0	0	0	5	=	-7
R_3	0	0	0	0	0	0	0	0	-1	0	3		-5
R_4	0	0	0	0	0	-1	0	0	0	0	3		-5
R_5	0	-1	0	0	0	0	0	0	0	0	5		-4
R_6	0	0	0	0	0	0	-1	0	0	0	1		-3
R_7	0	0	0	-1	0	0	0	0	0	0			-2
R_8	0	0	0	0	0	0	0	0	0	-1			-1

Table 7. Dual of First Feasible Solution in Matrix Equation Form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{bmatrix} = \begin{bmatrix} 10 \\ 15 \\ 8 \\ 12 \\ 7 \\ 11 \\ 9 \\ 5 \\ 2 \\ 6 \end{bmatrix}$$

Row 7: $u_3 - v_6 = 9$

Row 8: $u_4 - v_1 = 5$

Row 9: $u_4 - v_3 = 2$

Row 10: $u_4 - v_8 = 6$

Thus, we have a set of ten equations in 12 unknowns; the ten equations can be solved uniquely only if the values of two of the 12 unknowns are zero. As each $u_i \geq 0$ and each $v_j \geq 0$, we can discover which two of the unknowns are zero by the following reasoning:

(1) The right-hand side of each equation is positive; therefore, the v_j in each equation is smaller than the u_i in that equation; hence, $u_1 > 0$, $u_2 > 0$, $u_3 > 0$ and $u_4 > 0$.

(2) From Rows 1 and 2 it is clear that v_5 is smaller than v_1; hence, $v_1 > 0$.

(3) From Rows 3 and 4 it is clear that v_7 is smaller than v_2; hence, $v_2 > 0$.

(4) From Rows 6 and 7 it is clear that v_4 is smaller than v_6; hence, $v_6 > 0$.

(5) From Rows 9 and 10, it is clear that v_8 is smaller than v_3; hence, $v_3 > 0$.

The remaining v_j's two of which might have zero values are v_4, v_5, v_7 and v_8. We note the following additional points:

(6) From Row 2, $u_1 \geq 15$. Therefore,

(7) From Rows 1 and 5, $u_3 \geq 12$, and

(8) From Rows 1 and 8, $u_4 \geq 10$. Hence,

(9) From Row 6, $v_4 \geq 1$, and

(10) From Row 10, $v_8 \geq 4$. Therefore, only v_5 and v_7 may have zero values; $v_5 = 0$ and $v_7 = 0$.

Given $v_5 = 0$, we find from Row 2 that $u_1 = 15$; then, from Row 1, that $v_1 = 5$; from Row 5, that $u_3 = 12$; from Row 6, that $v_4 = 1$; and from Row 8, that $u_4 = 10$.

Given $v_7 = 0$, we find from Row 4 that $u_2 = 12$; from Row 3, that $v_2 = 4$; from Row 9, that $v_3 = 8$; from Row 7, that $v_6 = 3$; and from Row 10, that $v_8 = 4$. This completes the solution.

We next find that $c_{33} - u_3 + v_3 = 2$ and $c_{46} - u_3 + v_3 = 1$ so x_{33} will be increased. The best way to increase x_{33} involves increasing x_{33} by one unit, decreasing x_{43} by one unit, increasing x_{41} by one unit, and decreasing x_{31} by one unit.

A new dual solution is then computed. This indicates that x_{46} should be increased. After increasing x_{46} by three units, the solution (Table 8) is given by $x_{11} = 5$, $x_{15} = 4$, $x_{22} = 7$, $x_{27} = 2$, $x_{33} = 4$, $x_{34} = 5$, $x_{41} = 4$, $x_{43} = 1$, $x_{46} = 3$, and $x_{48} = 1$. The corresponding dual solution (Table 9) is $u_1 = 15$, $u_2 = 12$, $u_3 = 14$, $u_4 = 10$, $v_1 = 5$, $v_2 = 4$, $v_3 = 8$, $v_4 = 3$, $v_5 = 0$, $v_6 = 2$, $v_7 = 0$, $v_8 = 4$.[9/] At this point

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9/ Table 9 can be written out as ten ordinary algebraic equations: (footnote continued on page 35)

Table 8. Faculty Allocation Problem: Second Feasible Solution in Matrix Equation Form

Faculty Member (F_i) and Courses and Projects (R_j)	x_{11}	x_{15}	x_{22}	x_{27}	x_{33}	x_{34}	x_{41}	x_{43}	x_{46}	x_{48}	Units Assigned		Availabilities and Requirements
F_1	1	1	0	0	0	0	0	0	0	0	5		9
F_2	0	0	1	1	0	0	0	0	0	0	4		9
F_3	0	0	0	0	1	1	0	0	0	0	7		9
F_4	0	0	0	0	0	0	1	1	1	1	2		9
R_1	-1	0	0	0	0	0	-1	0	0	0	4		-9
R_2	0	0	-1	0	0	0	0	0	0	0	5	=	-7
R_3	0	0	0	0	-1	0	0	-1	0	0	4		-5
R_4	0	0	0	0	0	-1	0	0	0	0	1		-5
R_5	0	-1	0	0	0	0	0	0	0	0	3		-4
R_6	0	0	0	0	0	0	0	0	-1	0	1		-3
R_7	0	0	-1	0	0	0	0	0	0	0			-2
R_8	0	0	0	0	0	0	0	0	0	-1			-1

Table 9. Dual of Second Feasible Solution in Matrix Equation Form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{bmatrix} = \begin{bmatrix} 10 \\ 15 \\ 8 \\ 12 \\ 6 \\ 11 \\ 5 \\ 2 \\ 8 \\ 6 \end{bmatrix}$$

$c_{ij} \leq u_i - v_j$ for i = 1, 2, 3, 4, and j = 1, 2, 3, 4, 5, 6, 7, 8. Thus, an optimal solution has been obtained. The resulting faculty assignments are as follows:

Faculty Member	Assignments (Units): Course 1	2	3	4	5	6	Research 7	8	a_i
1	5				4				9
2		7					2		9
3			4	5					9
4	4		1			3		1	9
b_j	9	7	5	5	4	3	2	1	36

The value of the objective function, $W = \sum_{i=1}^{4} \sum_{j=1}^{8} c_{ij} x_{ij}$, is 321 in the optimal solution compared with 310 in the first feasible solution.

A. <u>Some Technical Aspects of the Objective Function Weights (c_{ij}'s)</u>

The c_{ij}'s in real situations would most likely be based on the judgment of the department chairman. Once specified, they guide the

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9/ (Continued from page 32)

Row 1: $u_1 - v_1 = 10$
Row 2: $u_1 - v_5 = 15$
Row 3: $u_2 - v_2 - v_7 = 8$
Row 4: $u_2 = 12$
Row 5: $u_3 - v_3 = 6$
Row 6: $u_3 - v_4 = 11$
Row 7: $u_4 - v_1 = 5$
Row 8: $u_4 - v_3 = 2$
Row 9: $u_4 - v_6 = 8$
Row 10: $u_4 - v_8 = 6$

Again we have ten equations in 12 unknowns. By the same reasoning as before we find that $v_5 = 0$ and $v_7 = 0$. We then solve the ten equations for the ten remaining unknowns, obtaining the values listed in the text.

allocation process to an optimal solution. But the chairman's judgment in specifying the c_{ij}'s may be fallible. It is worth considering, then, how sensitive the optimal set of assignments may be to variations in the c_{ij}'s. We can state the following points:

1. If all $m \cdot n$ of the c_{ij}'s are multiplied by the same constant, the optimal set of assignments will not be changed. For example, if all c_{ij}'s in Table 3 are multiplied by 0.5, $c_{15} = 7.5$ will be the highest weight in the resulting table; $c_{11} = 5$ will be the next highest weight for an activity involving Faculty Member 1; and so on. Activities x_{15} (at four units) and x_{11} (at five units) will be logical first steps toward a feasible solution, just as before. The new value of the objective function for the optimal solution will be $0.5W = 0.5(321) = 160.5$.

2. If the same positive constant is added to all $m \cdot n$ of the c_{ij}'s the optimal set of assignments will not be changed.[10/] For example, if we add two to every c_{ij} in Table 3, $c_{15} = 17$ will be the highest weight in the new table; $c_{11} = 12$ will be the next highest weight for an activity involving Faculty Member 1; and so on. The value of the new objective function for the optimal solution will be

$$W_{(+2)} = \sum_{i=1}^{4} \sum_{j=1}^{8} (c_{ij}+2)(x_{ij}) = \sum_{i=1}^{4} \sum_{j=1}^{8} c_{ij}x_{ij} + 2\sum_{i=1}^{4} \sum_{j=1}^{8} x_{ij}$$

$$= W + 2(36) = 321 + 72 = 393 .$$

10/ Instead of adding a constant to each c_{ij} we could subtract a constant provided that no c_{ij} is reduced below zero.

3. If all $m \cdot n$ of the c_{ij}'s are multiplied by the same constant β and are also all increased by the same constant α, the optimal set of assignments will not be changed. The new objective function will be

$$W_{(\beta,\alpha)} = \sum_{i=1}^{m} \sum_{j=1}^{n} (\beta c_{ij} + \alpha)\, x_{ij}$$

$$= \beta \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} + \alpha \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = \beta W + \alpha(36) \;.$$

In the present example, if $\beta = 0.5$ and $\alpha = 2$, the value of $W(\beta,\alpha)$ associated with the optimal set of assignments will be

$$W_{(\beta,\alpha)} = 0.5W + 2(36) = 0.5(321) + 72$$
$$= 160.5 + 72 = 232.5 \;.$$

In the resulting c_{ij} table, $c_{15} = 9.5$ will be the highest weight; $c_{11} = 7$ will be the next highest weight for an activity involving Faculty Member 1; and so on.

Thus, any linear transformation applied to all $m \cdot n$ of the c_{ij}'s will leave the optimal set of assignments unchanged, provided that no c_{ij} is reduced below zero by the transformation.

The reason for this rather encouraging stability of the optimal assignment set in the face of linear transformations or "codings" of the c_{ij}'s may be clarified by an illustration.

	Course 1	Course 2	Units Available (a_i)
Faculty Member 1	10	7	3
Faculty Member 2	8	6	3
Units Required (b_j)	4	2	6

The optimal solution is $x_{11} = 3$, $x_{12} = 0$, $x_{21} = 1$ and $x_{22} = 2$; $W = 3(10) + 0(7) + 1(8) + 2(6) = 50$.

What happens if we now transfer one unit of Faculty Member 1's time from Course 1 to Course 2? Clearly, we must transfer one unit of Faculty Member 2's time in the opposite direction, from Course 2 to Course 1. The "gain" in rearranging Faculty Member 1's time is (-10+7); the gain in rearranging Faculty Member 2's time is (-6+8). Thus, we lose 3 points on Faculty Member 1 and gain 2 points on Faculty Member 2; the net loss on the rearrangements is

$$(-c_{11}+c_{12}) + (-c_{22}+c_{21}) = (-10+7) + (-6+8) = (-3) + (2) .$$

The optimal solution is stable because any attempt to change it results in a loss-to-gain ratio of $-\frac{3}{2}$. If we add a constant, say 2, to each c_{ij}, we have (-12+9) + (-8+10) = (-3) + (2); the numerator and denominator of the loss-to-gain ratio are unchanged, so the ratio itself is unchanged. If we multiply each c_{ij} by a constant, say 0.5, we have (-5+3.5) + (-3+4) = (-1.5) + (1); the loss-to-gain ratio is still $\frac{-1.5}{1} = \frac{-3}{2}$, as before.

Thus, the stability of the optimal set of assignments depends on the stability of relative differences or loss-to-gain ratios associated with unit rearrangements; each rearrangement, as we have seen, involves four c_{ij}'s. Transformations of the type $c^*_{ij} = \alpha + \beta c_{ij}$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, where α and β are arbitrary constants, do not change the relative differences.

In the present example, the optimal assignment set would be stable under slightly less restrictive conditions. For example, we could multiply c_{11} and c_{12} by β and c_{21} and c_{22} by any constant strictly less than 1.5β (we assume $\beta > 0$) without changing the optimal solution. But it is hard to generalize when we go beyond uniform linear codings. A technique known as "sensitivity analysis" can, however, be used to determine the ranges of values over which stipulated c_{ij}'s may be varied without changing the optimal set of assignments.

B. Some Logical and Practical Aspects of the Objective Function Weights (c_{ij}'s)

Despite the technical points we have just discussed, it seems desirable to specify the c_{ij}'s as approximations to magnitudes which, in principle, could be given economic values and/or other values in the larger society. The vocational value of a college education is one of the most tangible of these magnitudes, and a good deal has been written on this subject by Schultz (18), Becker (2) and others. When a university president allocates funds between the professional schools and the College of Liberal Arts, some implicit judgments may be inferred--for example, the last million dollars allocated to Liberal Arts <u>should</u> be as productive (in terms of the president's value system or objective function) as the last million dollars allocated to the professional schools in which career income is an important and fairly predictable output of the training received.

If we value Faculty Member 1's contribution in a section of Course 5 at 15 points and in a section of Course 1 at 10 points, we ought to mean that we think he accomplishes $\frac{15}{10}$ as much "good" in Course 5 as in Course 1. If Faculty Member 2's contribution in a section of Course 5 is rated at 7 points, we ought to mean that we think he accomplishes $\frac{7}{15}$ as much "good" per section in that course as does Faculty Member 1. In a vocationally-oriented department, "good" may be roughly proportional to "increase in probable career income of students taking the course."

We might alternatively think of the c_{ij}'s as estimates of the national market values per course (i.e., the average salary cost per course) of professors who can teach course j as well as faculty member i. Would it cost about $15,000 to hire another professor who could teach Course 5 as well as Faculty Member 1? Would it cost about $10,000 to hire someone who could teach Course 1 as well as Faculty Member 1? Competition for faculty members does express itself in terms of salaries, teaching loads, class sizes and other considerations, most of which have a direct bearing on salary cost per course or per student quarter, so there would be some realism in trying to relate the c_{ij}'s to salary costs of hiring comparable performance in the national market. In general, it seems that the salary costs per section in different courses should be roughly proportional to the amounts of "good" done to the students, so the two approaches could lead to approximately the same set of c_{ij}'s.

For the moment, let us assume that the c_{ij}'s in Table 3 are estimates of the national average salary cost of obtaining the specified levels of performance in the stipulated courses. If $c_{15} = 15$, in other words, we assume Faculty Member 1 would justify a \$15,000 salary _if_ he were teaching nine sections of Course 5. (We leave aside the question of need for variety in one's teaching program.)

Given the size of the particular department, however, there are only four sections of Course 5 to be taught. The best use of Faculty Member 1's talents within this department is four units of Course 5 and five units of Course 1, and the average value of these services would be

$$\frac{4(15) + 5(10)}{9} = \frac{60 + 50}{9} = 12.222, \text{ or } \$12,222 .$$

If we apply this interpretation of the c_{ij}'s to all four faculty members we may summarize the results as follows:

Faculty Member	(1) Value as specialist in his best course (9 sections of it)	(2) Value if assigned to the 9 sections in which he has highest value to this department	(3) Value in optimal solution for department as a whole	(4) Value if used as in the first feasible solution
1	\$15,000	\$12,222	\$12,222	\$12,222
2	12,000	8,889	8,889	8,889
3	11,000	10,556	8,778	9,889
4	8,000	6,889	5,778	3,444
Totals:	\$46,000	\$38,556	\$35,667	\$34,444

In arriving at these figures, we have multiplied each c_{ij} by $\frac{1000}{9}$. If we reverse this procedure and multiply each of the Column (3) and Column (4) totals by $\frac{9}{1000}$ we have

$$35,667\left(\frac{9}{1000}\right) = 321$$

and

$$34,444\left(\frac{9}{1000}\right) = 310 .$$

The second result (310) will be recognized as the value of the objective function associated with the first feasible solution, while 321 is the value of the objective function associated with the optimal solution.

A few more comments are in order:

1. The optimal assignment set for each faculty member depends on the array of talents of all other faculty members in the department. Once Faculty Member 4 has been hired, the optimal use pattern for Faculty Member 3 is one valued at $8,778.

2. If Faculty Member 4 is on a one-year appointment while the other three members have tenure, in planning for the next following year it might be desirable to assign Faculty Member 3 to courses in which his total value is $9,889 (as in the first feasible solution) and try to recruit a new Faculty Member 4 who would be strong in Courses 1 and 3. (If Faculty Member 1 should leave, Faculty Member 3 could be assigned to courses in which his value is $10,556.)

3. Faculty Member 2 is evidently stronger in research than in teaching and might reasonably move to another institution which provides more time and facilities for research.

4. Faculty Member 1 has unusual qualifications for Course 5. These might extend to one or two closely related courses in the same field (perhaps at the first-year graduate level as well as at the advanced undergraduate level). A larger department with more enrollment in Course 5 and closely related courses could afford to offer Faculty Member 1 about $15,000.

5. The faculty allocation model maximizes an objective function pertaining to the department as a whole in a single year. Longer-run goals for the department could also be expressed as values of the objective function. Is it realistic to plan for a department in which the

average c_{ij} is 15 for the optimal assignment pattern? If so, the faculty would be worth (and would probably require) an average salary level of about $15,000, not counting the upward trend over time in the national salary structure for persons of given ability.

Faculty Member 1 is evidently of the desired quality, if used as a specialist. However, the peak performances of Faculty Members 2, 3 and 4 in their best courses are currently valued at $12,000, $11,000 and $8,000. Is this simply a matter of inexperience and other remediable factors? If not, the long-run goal for the department (average c_{ij} to equal 15) may be incompatible with the retention of some or all of these faculty members. Or, the goal might be redefined to state that <u>new</u> faculty members should be of the desired quality or potential (expected c_{ij}'s of 15); an average performance level of less than 15 for new and existing faculty members combined would be accepted as a fact of life during a fairly long transition period.

III. Models for Making Optimal Decisions Over a Sequence of Years

Decisions which educators make in a given year may affect next year's alternatives. Decisions made about admissions of new students this year may affect the number that can be admitted in subsequent years. Faculty members recruited this year will typically remain on the staff for several or many years, affecting program quality and other faculty recruitment needs and opportunities throughout their tenure.

A. Recursive Programming Models

Recursive programming models can be used to examine the effect of this year's decisions on future years's alternatives. They can also be used to show the sorts of decisions which might be made if only this year's information is used as the basis for making certain decisions.

The most commonly used recursive programming models have been patterned after the model of Richard Day (9). These models include recursive constraints on activity levels and on resource use which limit the increases (or decreases) in activity levels and resource use over last year's levels to certain percentages of last year's level. The objective function weights are often based on last year's market prices.

Recursive programming models of educational institutions might well include "flexibility constraints" to limit the extent to which this year's activity levels deviate from last year's activity levels since educators probably would not want admissions, staff additions, and so forth to vary widely from year to year. However, unlike the situations for which recursive programming models have ordinarily been designed, the production processes relevant to educational institutions often require more than one time period to complete. Thus the amount of teaching resources available for instructing this year's freshmen depends upon the amounts required to teach sophomores, juniors and seniors (new students admitted during the past three years). Thus a second sort of recursive constraint must be included in recursive models of educational institutions.

Including both flexibility constraints and the recursive constraints due to multi-period production can lead to infeasible solutions. In such a case it seems appropriate to require the recursive constraints arising from multi-period production to be satisfied and to allow, if necessary, the flexibility constraints to be violated.

In order to avoid this problem in the example which follows, the flexibility constraints will be eliminated and replaced with quadratic terms in the objective functions, W, which tend to favor (in a ceteris paribus sense) last year's activity levels. For example, a change of 20 percent in the level of activity i from the preceding year would involve a "penalty" (a subtraction from W) four times as large as the "penalty" for a 10 percent change; the penalty for a30 percent change would be nine times as large as that for a 10 percent change.

For the model to be used here the objective function weights will be recursively determined. Since educational institutions are not confronted with markets in which output prices vary widely from year to year, the same output "prices" will be used for all time periods. Because the productive processes in which educational institutions are involved require more than one period to complete there is a problem of allocating these "prices" among the several years required to produce the various outputs. More specifically the problem is one of deciding what portion of the output "price" to allocate to the first period of the production process. For the specific model to be used here the "shadow prices" from the previous year's solution will be instrumental in this allocation.

Both the "Day" type of recursive model and the type to be used here are presented in Appendix 2.

The specific problem to be considered is similar to that which might be faced by a department which is capable of awarding both Bachelor's and Master's degrees. In order to make the problem manageable certain simplifying assumptions will be made. Although in actual practice graduate students take some and undergraduate students take most of their courses in other departments, the model will assume that all courses are taken within the major department. It will also be assumed that the department can set admission levels for both graduate and undergraduate students.

Certain other assumptions will also be made. Undergraduate class sizes will be set at 35 students; graduate class sizes will be set at 18 students. Undergraduate students are assumed to take (on the average) 17.5 courses per year for four years; M. S. students are assumed to take 9 courses during the first year of their studies and 3 courses plus 9 credit hours of research during their second year of study. It is assumed that supervising 18 credit hours of thesis credit requires as much teaching resources as is required to teach one graduate course. It is assumed that the department has six faculty members, that each of these faculty members has signed a contract calling for supplying teaching inputs equivalent to that required to teach 8 sections and that the permitted division between undergraduate and graduate teaching ranges between 7 undergraduate sections and 1 graduate section per year per faculty member to 4 undergraduate sections and 4 graduate sections per faculty member per year. It is also assumed that the teaching budget for each year includes 2 positions for graduate teaching assistants who each supply 5 units of teaching inputs per year. It is assumed that for the courses taken by freshmen and sophomores up to 40 percent of the teaching inputs can be supplied by teaching assistants without loss of instructional quality. The department is assumed to be limited to 30 new freshmen per year and 7 new graduate students. That is, these are assumed to be the maximum numbers of new students willing to enroll each year. The department is assumed to assign relative prices of 3 to 2 to B.S. and M.S. degrees.

The actual activities and constraints used are presented in Appendix 2.

It was assumed that at the beginning of the first year considered by the model the department had 21 seniors, 22 junirs, 23 sophomores and 6 graduate students.

The approximate admissions solution generated by the model are presented in Table 10. These solutions are characterized by a period of adjustment from period 1 through period 10 followed by 4-year cycles from period 10 onward. The admissions solutions are shown in more detail in Table 11, along with the calculations of the value of the objective function in each year.

Table 10. Optimal Admissions of Undergraduate and Graduate Students Respectively as Computed from a Recursive Programming Model

Year	New Undergraduates	New Graduate Students
1	30	7
2	28	3
3	23	5
4	20	5
5	21	7
6	22	7
7	23	7
8	24	7
9	25	7
10	26	7
11	24	7
12	24	6
13	25	7
14	26	7
15	24	7
16	24	6
17	25	7
18	26	7
19	24	7
20	24	6
21	25	7

Table 11. A Recursive Programming Model to Optimize Undergraduate and Graduate Student Admissions

	Admissions: Undergraduate				Graduate		Objective Function Calculations:								Objective Function Value:
Year	x_{1t-3}	x_{1t-2}	x_{1t-1}	x_{1t}	x_{2t-1}	x_{2t}	$3x_{1t}$	$+ 2x_{2t}$		$- 0.2(x_{1t}-x_{1t-1})^2$		$- 0.2(x_{2t}-x_{2t-1})^2$		=	W
1	21	22	23	30	6	7	90	+	14	- 9.8	-	0.2		=	94.0
2	22	23	30	28	7	3	84	+	6	- .8	-	3.2		=	86.0
3	23	30	28	23	3	5	69	+	10	- 5.0	-	.8		=	73.2
4	30	28	23	20	5	5	60	+	10	- 1.8		0		=	68.2
5	28	23	20	21	5	7	63	+	14	- .2	-	.8		=	76.0
6	23	20	21	22	7	7	66	+	14	- .2		0		=	79.8
7	20	21	22	23	7	7	69	+	14	- .2		0		=	82.8
8	21	22	23	24	7	7	72	+	14	- .2		0		=	85.8
9	22	23	24	25	7	7	75	+	14	- .2		0		=	88.8
10	23	24	25	26	7	7	78	+	14	- .2		0		=	91.8
11	24	25	26	24	7	7	72	+	14	- .8		0		=	85.2
12	25	26	24	24	7	6	72	+	12	0	-	.2		=	83.8
13	26	24	24	25	6	7	75	+	14	- .2	-	.2		=	88.6
14	24	24	25	26	7	7	78	+	14	- .2		0		=	91.8
15	24	25	26	24	7	7	72	+	14	- .8		0		=	85.2
16	25	26	24	24	7	6	72	+	14	- .8		0		=	85.2
17	26	24	24	25	6	7	75	+	14	- .2	-	.2		=	88.6

It is apparent that the recursive solutions are very efficient from period 10 or so onward. Some idea of the relative inefficiency of this particular recursive model can be obtained by comparing its solutions during the first 10 periods to the solution of a dynamic programming model covering the same interval of time.

B. Dynamic Programming Models

In order to make the comparison meaningful the dynamic programming model was required to satisfy the same initial conditions as the recursive model and in addition it was required to leave sophomore, junior, senior and second-year graduate student enrollments in period 11 at the same levels as those generated by the recursive model.

The solution to the dynamic programming model is presented in Table 12. Since the dynamic programming model maximizes a single objective function for the whole 10-year period rather than sequentially maximizing an objective function for each year it was to be expected that its performance would be somewhat better than the recursive programming model. The dynamic programming model allows total admissions of at least 250 undergraduate students and 61 graduate students. This is 8 more undergraduate students and 1 fewer graduate student than allowed by the recursive programming model during the same ten-year period.

There is little doubt that if resource availabilities can be predicted accurately for several years into the future a dynamic programming model will be better for planning purposes than a recursive programming model. On the other hand, if resource levels can be predicted accurately

Table 12. Optimal Admissions of Undergraduate and Graduate Students as Computed from a Dynamic Programming Model

Year	New Undergraduates	New Graduate Students
1	30	7
2	24	7
3	23	5
4	23	6
5	24	3
6	25	7
7	26	7
8	24	6
9	25	6
10	26	7

only for the current year (or perhaps only two or three years at a time) the relative advantage of dynamic programming models is greatly diminished.

The specific recursive programming model considered here proved to be somewhat more efficient than had been expected. It seems likely that decision models based on similar recursive programming models will be far less efficient than would be indicated by the results presented in Table 10. The need to make decisions recursively (i.e., once each year or once each planning period) can hardly be avoided since the accuracy or certainty of the information available about resource availabilities and so forth for any particular time period is likely to increase as that time period draws nearer. However, it would seem reasonable to expect that better decisions could be made this year if whatever information is available about future years is used even if this information is not known with certainty. Such an approach would probably require the determination (tentatively) of admissions (and other activity) levels for future periods as well as for the current period. The future admission levels would of course need to be revised the following year if new information became available. It is likely that such an approach could also take advantage of some of the techniques of stochastic programming.

C. A Common Sense Interpretation of the Dynamic Programming Model

All the values of variables involved in the ten-year dynamic programming model are shown in Table 13. The model itself in matrix equation form is presented in Table 16, Appendix 2.

2

Table 13. Dynamic Programming Model to Optimize Undergraduate and Graduate Admissions over a 10-Year Period

Item		YEAR 1	2	3	4	5	6	7	8	9	10
Variables to be optimized: (computed values rounded down if not integers)	x_{1t}	30	24	23	23	24	25	26	24	25	26
	x_{2t}	7	7	5	6	3	7	7	6	6	7
Other variables:	x_{1t-3}	21	22	23	30	24	23	23	24	25	26
	x_{1t-2}	22	23	30	24	23	23	24	25	26	24
	x_{1t-1}	23	30	24	23	23	24	25	26	24	25
	x_{2t-1}	6	7	7	5	6	3	7	7	6	6
	x_{3t}	50	50	48	47	12	50	50	50	49	50
	x_{4t}	3	5	0	0	36	0	2	0	0	1
	x_{5t}	4	4	2	1	1	0	4	4	3	3
Shadow prices corresponding to restrictions: (App. 2, p.	(23)	0	0.85	0.40	0.60	0.75	0.85	0.40	0.91	0.45	0.94
	(24)	0	0.85	0.67	1.00	0.75	1.25	0.40	1.00	0.75	0.94
	(25)	0	0.28	0.22	0.33	0.25	0.31	0.13	0.33	0.25	0.31
Objective function weights:	c_{1t}	3.00	3.00	3.00	3.00	3.00	3.00	3.00	(3.00) 1/	(3.00) 1/	(3.00) 1/
	c_{2t}	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	(2.00) 1/
Objective function Value:											

$W = 644.19$ 1/

1/

Freshmen admitted in Years 1 through 7 receive B.S. degrees in Years 4 through 10 and enter the objective function; freshmen admitted in Years 8, 9 and 10 do not enter objective function.

Graduate students admitted in Years 1 through 9 receive M.S. degrees in Years 2 through 10; those admitted in Year 10 do not figure in the objective function value.

The complete model involves 50 activities (5 activities in each of ten years) and 30 restrictions (3 restrictions in each of ten years). The activity levels in, say, Year 3 depend partly on the numbers of freshmen admitted in Years 0, 1 and 2 and the number of beginning graduate students admitted in Year 2. The activities and restrictions <u>directly</u> relevant to Year 3 are as follows:

Relevant Activities:

	Years 0, 1 and 2:				Year 3					
	BS0	BS1	BS2	MS2	BS3	MS3	I3	T3	TR3	
c_{it}	*	*	*	*	3.00	2.00				
J3			1.0		1.0		-1.0	-1.0		≤ 0
U3	1.0	1.0					0.6	1.0	1.0	≤ 84
G3				4.0		3.0			-3.0	≤ 36
X_{it}	23	30	24	7	23	5	48	0	2	

$W_3 = 23\ (3.00) + 5\ (2.00) = 69.00 + 10.00 = 79.00$

* Not used in calculating W_3.

The objective of the department is to convert 48 sections' worth of faculty time each year (over the ten-year period) into as many "output points" as possible, given that each B.S. degree is valued at 3 points and each M.S. degree at 2 points. (As the M.S. candidates already have the B.S. degree, the 2 points represent an increment of value added over and above the B.S. degree).

There is only one activity available for producing M.S. degrees but there are two alternative activities for producing B.S. degrees. The cost of a unit level of each activity in sections of faculty time is:

M.S.: 1.17 sections of faculty time

B.S.(I): 2.00 sections of faculty time

B.S.(II): 1.60 sections of faculty time.

(The B.S.(II) activity also requires 0.40 sections of teaching assistant time, but we will disregard the cost of this input for our present purpose).

The value of output per section of faculty time in each activity is:

M.S.: $\frac{2.00 \text{ points}}{1.17 \text{ sections}}$ = 1.70 points per section

B.S.(I): $\frac{3.00 \text{ points}}{2.00 \text{ sections}}$ = 1.50 points per section

B.S.(II): $\frac{3.00 \text{ points}}{1.60 \text{ sections}}$ = 1.88 points per section.

Within the terms of the problem, Activity B.S.(II) produces the greatest value per section; Activity M.S. is second and Activity B.S.(I) is third.

Of the three restrictions in each year, J insures that freshmen and sophomores get the required amount of instruction, U that juniors and seniors get the required amount of instruction, and G that graduate students get the required amount of instruction and thesis supervision. The unit of measure in restrictions J and U is one-half sections of faculty time; the unit for G is one-sixth section of faculty time.

In Table 13, the shadow prices corresponding to each restriction in a given year indicate the number of points by which the value of the objective function could be increased if one more unit of the restricting resource were available (one more half-section in J and U, one more sixth-section in G). The difference in units is inconvenient, so we multiply the shadow prices of J and U by 2 and that of G by 6 to obtain the increase in value of the objective function <u>per additional section of faculty time</u> in all three cases with the following results:

Year	Shadow Prices Per Section of Faculty Time: J	U	G	Levels of Selected Activities x_3(=I)	x_4(=T)	x_5(=TR)
1	0	0	0	50	3	4
2	1.70	1.70	1.70	50	5	4
3	0.80	1.33	1.33	48	0	2
4	1.20	2.00	2.00	47	0	1
5	1.50	1.50	1.50	12	36	1
6	1.70	2.50	1.86	50	0	0
7	0.80	0.80	0.80	50	2	4
8	1.82	2.00	2.00	50	0	4
9	0.90	1.50	1.50	49	0	3
10	1.88	1.88	1.88	50	1	3

The basic economic problem is to allocate 48 sections of faculty time among freshman-sophomore, junior-senior and graduate level teaching so as to maximize the value of total output. The shadow prices are marginal value products; in the continuous cases usually stressed in economic theory, the value of an additional section of faculty time should be the same in all three uses.

In Years 1, 2, 5, 7 and 10 this three-way equality applies. In Years 3, 4, 8 and 9 the shadow prices of U and G are equal but the shadow price of J (marginal value product of faculty time in freshman and sophomore teaching) is lower. In Years 3, 4, and 9, the shadow price of J is 0.6 times as large as the shadow price of U.

Activity x_5 has the effect of equating the shadow prices of U and G whenever $x_5 > 0$, and x_5 is greater than zero in all years except Year 6. Activity x_4 has the effect of equating the shadow prices of J and U when $x_4 > 0$, as it is in Years 1, 2, 5, 7 and 10. When Activity $x_4 = 0$, in most (but not all) cases the marginal value product of faculty time in teaching freshmen and sophomores is only 60 percent as large as that in teaching juniors and seniors under the assumptions of our problem.

In Year 2, it appears that the most "profitable" activity to expand is the production of M.S. degrees; in Year 5 the production of B.S. degrees using Activity B.S.(I) without teaching assistant help; and in Year 10 the production of B.S. degrees using Activity B.S.(II) with teaching assistant help.

Evidently the solutions of dynamic programming models can be given common sense interpretations. The technique of solution is essentially that used for ordinary linear programming models. The total numbers of activities and restrictions increase with the number of years in the planning period and also with the real complexity of the department and/or the degree of detail with which it is represented. The Plessner-Fox-Sanyal model (17) has 15 activities and 16 restrictions in each year of a four-year planning period, or 60 activities and 64 restrictions for the period as a whole.

The McCamley model (15) of a large Economics department has 82 activities and 57 constraints for a single year; a ten-year dynamic programming version at this level of detail would have 820 activities and 570 restrictions. A model of this size would not exceed the capacity of modern computers; however, it remains to be seen whether this level of detail is needed (or is in some sense useful) in a ten-year planning model.

IV. A Two-Level Decision Model for Allocating the Resources of a College Among Its Constituent Departments

The third problem to be discussed is that of allocating fixed resource supplies among several departments (or among other types of suborganizations or sectors). The dean of a college may be granted a certain budget for the operation of his college. He is also likely to have resources such as office space and classroom space which are available to him in limited quantities. He may want to allocate these resources so as to achieve a maximum value of some objective function.

In addition to the resources which obviously need to be allocated there will usually be other products whose use and production must be coordinated. In the previous model it was assumed that students take all of their courses in one department. This, of course, is not the case. There is, therefore, a need to insure that the amounts of instruction required by students outside their own departments do not exceed the amounts supplied by the various departments which supply service teaching. In a planning model it may therefore be appropriate to consider the dean as being concerned with the allocation of that instruction and of those other outputs which are produced by one department for use by other departments within the same college.

One way for the college dean to decide on the appropriate allocations of the various resources would be for him to treat his college as one large decision-making unit. He could decide how much of each output to produce and what input combination to use in producing it.

A byproduct of these decisions would be decisions about the amount of each resource to allocate to each department.

Fortunately, in some cases the dean may be able to allocate resources among departments, let the departments make most of the output and other activity level decisions, and still accomplish whatever output objectives he may have in mind. One of these cases occurs when all of the constraints are linear, the college objective function is linear, and when in addition, the departmental objective functions assign the same weights to the various outputs and inputs as are assigned by the college objective function. The results of Kornai and Liptak (13) assure us that in such a case there exists a system of quotas (for the resources allocated by the college dean) that, if implemented, will insure that the college dean's objective function will be maximized. The decomposition algorithm of Dantzig and Wolfe (8) provides a basis for the construction of a decentralized decision-making approach which can be used to discover an optimum set of quotas. Under this approach the college dean would, at each phase, ask each department how much of each resource it could "profitably" use if certain "prices" were assigned to each resource. The information which the departments give him would be used to aid in the derivation of a new set of "prices." The college dean would then ask each department to tell him how much of each resource it would use at these new prices. This process would continue until an optimum set of quotas is obtained.

A more precise description of this approach and its termination conditions can be found in the appendix. The primary advantage of this approach (and of others like it) is that it shifts much of the decision-making responsibility from the college dean to the individual departments.

Consider a college having three departments (departments A, B, and C). Suppose that the college has a teaching budget of 220 thousand dollars per year, that 80 (graduate) students from other colleges take courses in department A, 115 take courses in department B, and 210 take courses in department C. Suppose further that graduate students in departments B and C take courses in department A and that graduate students in department A take courses in department B. To simplify matters it will be assumed that undergraduate enrollment, curriculum, and distribution of undergraduate students among majors is predetermined.

Some of the assumptions which will be made concerning the individual departments are outlined in Table 14. The specific models used for the departments are presented in Appendix 3. These models all allow alternative input combinations in the production of research publications, and in the teaching of undergraduate students. They classify graduate students according to means of support. They also permit varrying amounts of research, graduate teaching, and undergraduate teaching per faculty member.

The specific formulations used are presented in Appendix 3.

The model required 7 phases to reach an optimum set of quotas. The "prices" and corresponding resource use levels for each phase are presented in Appendix 3, Table 21. An optimum set of quotas is also presented in Appendix 3, Table 21. The departmental solutions are presented in Appendix 3, Table 22.

Table 14. Two-Level Decision Model: Some Characteristics of Departments A, B & C

Item	Department A	Department B	Department C
No. of faculty members [a]	5.5	8.5	7.5
Undergraduate teaching required (no. of student courses)	1850	2750	2250
Graduate service teaching required (no. of student courses)	determined by college dean		210
Research budget ($'s)	40,000	20,000	45,000
Faculty salaries ($'s)	11,000	10,000	12,000
Undergraduate class sizes			
Faculty instructors	35	30	40
Both faculty and graduate student instructors	30	25	35
Graduate class sizes	24	15	18
Thesis "class sizes"	6.5	7.0	7.5
Teaching Assistant Salaries ($'s)	2800	2600	2750
Research Assistant Salaries ($'s)	2750	--	2700
Number of inputs supplied by teaching assistants (sections taught per year)	5	5	6
Number of years required to obtain M.S. degree	2	2	2
Number of courses taken to obtain M.S. degree			
in: Department A	10	3	4
in: Department B	3	10	0
in: Department C	0	0	9
Thesis Credits	3	3	3
Objective function weights research publications	2.50	2.00	3.00
M.S. degrees	1.50	1.75	2.00

a/ Each department is assumed to have an integral number of faculty members one of whom devotes half of his time to administrative functions.

The college model as a whole looks fairly complicated. It is important, therefore, to consider the model for one of the departments in considerable depth. This will assure that we understand the logical structure of the model, the realism of its content (class sizes, teaching loads, and the like), and the mechanisms through it allocates limited resources among the various departmental activities.

A. <u>A Common Sense Interpretation of the Model for Department A</u>

In Table 15 we have rearranged the rows and columns of the Department A model to emphasize the <u>relative</u> independence of its three major programs, research, undergraduate teaching and graduate teaching, including thesis supervision.

1. <u>Sizing up the research program</u>. When this is done, we see that the research program includes only three possible activities (A_1, A_2, A_3) and uses only three kinds of resources, namely research assistants (Row 4), faculty man years (Row 5), and research current expense funds (Row 6, in part). Activities A_5, A_{11}, A_{12} and A_{13} show that the salaries of research assistants and the research portions of faculty salaries, as well as research current expense funds, must all be fitted into the $40,000 research budget indicated in Row 6 of the Department restrictions column.

Regardless of anything else, then, our research program must stay within the $40,000 limit. The dollar costs of a unit of each research activity (that is, one man year of faculty time, plus research assistants, if any, plus current expense funds) are as follows:

Table 15. Department A Model: Rows and Columns Arranged to Emphasize the Relative Independence of Its Three Major Programs (Research, Undergraduate Teaching and Graduate Teaching)

															Restrictions Applying to Dept. A	Shadow Price in College Model Solution
Activity Levels 1/ (x_{ij}):	1.47	0.50	0.0	0.0	61.67	1.66	2.45	3.08	8.18	3.32	4.83	0.46	0.21			
Objective Function Weights (c_{ij}):	5.00	3.75	2.75			1.50	1.50	1.50								
Row No.	A1	A2	A3	A7	A8	A4	A5	A6	A9	A10	A11	A12	A13			
4	3	1					-2							≤	0	0.491
5	1	1	1								-1/3	-1/3	-1	≤	0	2.991
6	3000	1500	750				5500				3600	3600	11,000	≤	$40,000	0.178
7				-35	-30									≤	- 1,850	0.037
9				1	0.5						-6	-4		≤	0	1.423
11					0.5			-10						≤	0	0.810
1						10	10	10	-24					≤	UA1*	0.059
2						3	3	3						≤	UA2*	0.084
8						3	3	3		-6.5				≤	0	0.219
13						1								≤	2.0	0
10									1	1	-2	-4		≤	0	1.423
3								5600			7400	7400		≤	UA3*	1.446
12											1	1	1	≤	5.5	1.037

1/ Optimal levels in college model solution.
* Restrictions for which optimal values (quotas) are to be determined by the college dean.

Row No.	Resource	Cost per Unit of Resource (dollars)	Cost per unit of Activity: A1 (dollars)	A2 (dollars)	A3 (dollars)
4	Research assistants	2,750	8,250	2,750	0
5	Faculty time	11,000	11,000	11,000	11,000
6	Current expense	1	3,000	1,500	750
Total dollar cost (Row 6):			22,250	15,250	11,750

The outputs of a unit of each activity are valued as indicated by the objective function weights, the c_{ij}'s, at 5.00, 3.75 and 2.75 respectively for Activities A1, A2 and A3. The $40,000 budget restriction would permit a maximum of 1.798 units of Activity A1, or 2.623 units of Activity A2, or 3.404 units of ActivityA3. The values of the total outputs associated with each of these choices are given in Column (3) below:

Activity No.	(1) Maximun No. of Units of the Activity	(2) Value of Output per Unit of Activity	(3) Value of Maximum Output Col.1xCol.2	(4) Faculty Man-years required for maximum Output	(5) Dollar Cost Per Unit of Value of output: $40,000 ÷ Col. 3 (dollars)
A1	1.798	5.00	8.990	1.798	4,449.
A2	2.623	3.75	9.836	2.623	4,067.
A3	3.404	2.75	9.361	3.404	4,273

If we regard dollars as our most limiting factor, then Activity A2 is moderately superior to Activity A3 and Activity A3 is moderately superior to Activity A1. However, the differnces are not large; as indicated in Column (5) the dollar costs per unit of output-value are $4,067, $4,273 and $4,449. Presumably, a unit of output-value represents some combination of quantity, quality and importance of the research results obtained and reported. For example, an article of average importance in the refereed national journals most relevant to Department A might be rated at (say) 2.00 output-value units and the values of other kinds of research reports and articles could be related to this as a base. If the chairman of Department A has some doubts about the precision of his c_{ij}'s, he will be somewhat diffident about his ability to choose between Activities A1, A2 and A3 on the basis of Column (3) or its equivalent, Column (5). However, if dollars are the only scarce resource, the computer will (quite correctly on the basis of the numbers we feed into it) tell us to put our entire $40,000 into 2.623 units of Activity A2.

For future reference we must take note of the fact that Activity A1 is far superior to Activity A2 in terms of output per faculty man year (5.00 versus 3.75 value units) and Activity A2 to Activity A3 (3.75 versus 2.75 value units). Hence, if faculty time turns out to be the most limiting factor, we will be wise to emphasize Activity A1.

2. Undergraduate teaching. Activities A7 and A8 are the only ones involved directly in undergraduate teaching. Row 7 tells us that Department A must provide at least 1850 student courses of undergraduate instruction. Although no c_{ij}'s (objective function weights) are assigned to Activities A7 and A8, the restriction in Row 7 really gives undergraduate teaching an absolute priority over research and graduate teaching. Row 7 is a categorical imperative: "Thou shalt teach at least 1,850 student courses to undergraduates, regardless of other considerations." Alternatively, we could assign relatively high c_{ij}'s to Activities A7 and A8, to insure that the computer rated them as having higher values per faculty man year and

per dollar than any other activities. We would still need to specify an upper limit, presumably 1,850, on the number of student courses to be taught, or the computer might tell us to teach more student courses than any realistic estimate of the enrollment which will be forthcoming.

Row 7 says, then, that undergraduate teaching comes first. Activities A7 and A8 have the following meaning:

Row No.	Outputs (-) and Inputs (+)	Activity Number A7	Activity Number A8
7	Class size	-35	-30
	Proportion of teaching done by:		
9	Faculty	1	0.5
11	Teaching assistants	0	0.5

The basic unit here is the individual class. The salary cost of one class of 35 taught wholly by a faculty member (Activity A7) is $\frac{\$7,400}{8} = \$925$. The salary cost of an instructional pattern which uses 0.5 "sections' worth" of faculty time and 0.5 "sections' worth" of teaching assistant time per 30 students (Activity A8) is:

$$\text{Faculty:} \quad 0.5 \ \frac{\$7,400}{8} = \frac{7,400}{16} = \$462.50 \quad \text{Plus:}$$

$$\text{Teaching Assistant:} \quad 0.5 \ \frac{\$2,800}{5} = \frac{2,800}{10} = 280.00$$

$$\text{Total:} \quad \$742.50$$

Activity A7 costs $\frac{\$925}{35} = \$26.43$ per student course, while Activity A8 costs $\frac{\$742.50}{30} = \$24.75$ per student course. So far as the model is concerned, the two activities are equally acceptable in terms of quality per student-course.

The total resource costs of teaching 1,850 undergraduate student-courses by each method are:

	Activity A7	Number A8
Total output (in student-courses)	1850	1850
Number of units of activity required	52.86	61.67
Sections' worth required from:		
Faculty	52.86	30.83
Teaching assistants	0	30.83
Total dollar cost:	$48,896.	$45,788.

The dollar costs are moderately lower for Activity A8. Activity A8 is much more economical in the use of faculty time, if that proves to be a major consideration. Also, Activity A8 provides support for some graduate students.

3. <u>The graduate (M.A. or M.S.) program</u>: Activities A4, A5 and A6 have the following interpretations:

Row No.	Description	A4	A5	A6
		Activity Number		
*	Output: M.S.degrees in Dept. A.	1	1	1
1	Courses per student taken in Department A:	10	10	10
2	Courses per student taken in Department B:	3	3	3
8	"Equivalent courses" represented by thesis credit taken in Department A:	3	3	3
	Means of support:			
13	Not supported by university funds:	1		
4	Research assistants		-2	
6	Research budget		$5500	
11	Teaching assistants			-10
3	Teaching budget			$5600

* The outputs (one M.S. degree) are not stated explicitly in a single row but are implicit in Rows 13, 4, 6, 11 and 3.

The M.S. program is assumed to take two years regardless of the student's means of support. In the Activity A5 column, the -2 and the $5500 imply that he serves as a research assistant for two years on a part-time salary of $2,750 a year. Under Activity A6, the -10 and the $5600 imply that he teaches 5 classes a year for two years on a part-time salary of $2,800 a year.

Activities A9 and A10 have the following meanings:

Row No.	Description	Activity Number A9	A10
1	Class size	-24	
8	Thesis supervision, equivalent class size		-6.5
10	Sections' worth of faculty time	1	1
	Salary cost per equivalent section:	$ 925	$ 925
	Per equivalent student course:	$ 38.54	$142.31

It is assumed that each M.S. degree requires both the 10 courses in Row 1 and the thesis credit (equivalent to 3 courses) in Row 8; each of these activities draws directly on only one resource, namely faculty time (Row 10). It would therefore be logically possible to combine activities A9 and A10 to say that 10 actual courses plus 3 courses' worth of thesis credit require $\frac{10}{24} + \frac{3}{6.5} = 0.4167 + 0.4615 = 0.8782$ sections' worth of faculty time. At the same time Rows 1 and 8 could be combined to say that an M.S. degree requires 13 equivalent courses at an average equivalent class size of 14.8 students.

4. The "faculty allocation activities." Activities A11, A12 and A13 represent three different kinds of time allocations for faculty member, along with the corresponding charges against teaching and research budgets (Rows 3 and 6) and against the initial "pool" of faculty man years--in this case 5.5 (Row 12).

The items with negative signs in Rows 5, 9 and 10 of Activity A11 have the following meanings; per unit of Activity A11:

Row 5: One-third of a man year of faculty time is made available for research activities.

Row 9: Six sections of faculty time are made available for undergraduate teaching.

Row 10: Two sections of faculty time are made available for graduate teaching, including thesis supervision.

Eight sections a year are regarded as a two-thirds time load, leaving one-third time available for research.

A unit of Activity A 12 provides one-third of a faculty man year for research, four sections of time for undergraduate teaching, and four sections for graduate teaching. A unit of Activity A13 supplies one faculty man year exclusively for research.

5. Putting things together. Activities 11 and 12 imply that no faculty member shall be required to teach more than 8 courses a year. With only 5.5 faculty man years available, a maximum of 44 sections can be taught by faculty members. If Activity A11 were used exclusively, the maximum number of undergraduate sections to be taught by faculty members would be 5.5 times 6, or 33.

Recall that the undergraduate teaching activities A7 and A8 would require 52.86 and 30.83 sections of faculty time respectively. The model clearly requires us to use Activity A8 exclusively, or nearly so, with half of the total teaching time supplied by teaching assistants.

We must use at least 30.83 sections of faculty time in undergraduate teaching. If Activity A12 were used exclusively, we would have only 22 units available at the undergraduate level, so we will evidently have to rely largely on Activity A11, which can supply as many as 33.

Tentatively, then, we can decide to use Activity 8 exclusively for the 1850 student-courses of undergraduate teaching. This requires 30.83 sections of faculty time.

If we supply these 30.83 sections exclusively with Activity A11, we require $\frac{30.83}{6} = 5.4$ units of this activity.

Activity A11 at the level of 5.4 units provides $\frac{5.14}{3} = 1.71$ man years of faculty research time. As it seems clear that faculty time is a scarce resource, we may decide tentatively to use Activity A1 exclusively in our research program--1.71 units of it. At $22,250 per unit this would use up $38,048 of the $40,000 available for research.

At this point, we have 0.36 man years of faculty time unassigned and have done nothing at all about graduate teaching, though our tentative research program calls for 3(1.71) = 5.13 (or 5.14) research assistants.

We must recall that 5.14 units of Activity A11 provides 2(5.14) or 10.28 sections of faculty time for graduate teaching. Each M.S. degree requires 0.4167 sections of faculty time in course work and 0.4615 in thesis supervision, or 0.8782 in all. With 10.28 sections we can accommodate

$\frac{10.28}{0.8782} = 11.71$ M.S. degrees per year.

How does this estimate square with the following facts?

a. In choosing Activity A8 for undergraduate teaching, we expressed a need for 30.83 sections of help from teaching assistants at 5 sections per part-time assistant per year, or $\frac{30.83}{5} = 6.16$ teaching assistants.

b. In choosing Activity 1 for research, we expressed a need for 5.14 research assistants.

Thus, we require 6.16 plus 5.14 or 11.30 assistants for these two activities. As each one spends two years on the M.S. program, this group would lead to $\frac{11.30}{2} = 5.65$ M.S. degrees a year.

c. Activity A4 provides M.S. training for students not supported on university funds, but Row 13 restricts their number to not more than two M.S. degrees a year.

Hence, Activities A4, A5 and A6 combined could amount to not more than 7.65 M.S. degrees under our first round of decisions on activity levels. So, we appear to need only 7.65 (0.8782) or 6.72 sections of faculty time for graduate teaching of M.S. candidates in Department A instead of the 10.28 sections made available by 5.14 units of Activity A11.[11]

At this stage we will have arrived at the following value of Department A's objective function:

Research: 1.71 units of Activity A1 times 5.00 points per unit = 8.55 points
M.S. degrees: 7.65 degrees times 1.50 points per degree = 11.48 points
20.03 points.

6. Taking a second look. We probably have not yet reached the maximum possible value of the objective function, for the following reasons:

(1) We have \$40,000 - \$38,048 = \$1,952 of research funds unused;

(2) We have 0.36 man years of faculty time unallocated, which could evi-

11/ The solution of the college model (see Table 15) indicates that 8.18 sections are needed for teaching graduate courses and 3.32 sections for thesis superdently go into full-time research; and

(3) With 5.14 units of Activity A11, we have a little more capacity for graduate teaching than we need (for M.S. candidates in Department A only).

Evidently we could at least accomplish a little more research with our remaining resources.

So far we have made no allowance for the three restrictions on Department A's activities which are to be derived from the interaction between the dean and all three department chairmen (of Departments A, B and C).

Department A's teaching budget must evidently include the following:

Faculty: 5.14 times $7,400	= $ 38,036
Teaching assistants: 6.167 times $2,800	= $ 17,267
Total:	$ 55,303

We cannot determine the amount of service teaching required from Department A by the other departments, of course, without going through an approximate analysis of the models for those departments and ultimately for the college as a whole.

It will be worthwhile to compare the final results for Department A as part of the complete system with our preliminary common sense results:

11/ (continued)

vision, a total of 11.50 sections. But note that the college model requires Department A to teach 80 student courses, or $\frac{80}{24}$ = 3.33 sections, to graduate students from other colleges and also to supply service courses to some graduate students in Departments B and C.

Activity No.	Activity Level in Preliminary Analysis	Final Results of College Model: Activity Levels	Objective Function Weights	Contribution to Value of Objective Function
A1	1.71 to 2.07	1.47	5.00	7.35
A2	0	0.50	3.75	1.88
A3	0	0	2.75	0
A7	0	0	0	0
A8	61.67	61.67	0	0
A4	2.00	1.66	1.50	2.49
A5	2.57	2.45	1.50	3.67
A6	3.08	3.08	1.50	4.62
A9	Not comparable	8.18	0	
A10	3.53	3.32	0	
A11	5.14	4.83	0	
A12	0	0.46	0	
A13	0 to 0.36	0.21	0	
Total value of objective function:				20.01

Apparently, the college-level restrictions we ignored in our preliminary treatment of Department A approximately offset the gains we should have made by using up our remaining $1,948 of research budget and 0.36 man years of faculty time in the absence of college-level restrictions and interactions.

7. Meaning of the "shadow prices" in the final solution. In general, the shadow-price associated with any limited resouce is the amount by which the value of the objective function could be increased, directly and/or indirectly, if we had one more unit of that resource.

The shadow prices associated with the restrictions in the respective rows of Table 15 for Department A (in the final results) are as follows:

Restriction Contained in Row Number	Shadow Price	Unit in Which Restriction is Stated
4	0.491	Number of research assistants
5	2.991	Man years of faculty time
6	0.178	$1,000 of research funds
7	0.037	
9	1.423	One section of faculty teaching time
11	0.810	
1*	0.059	
2*	0.084	
8	0.219	
13	0	Restriction not effective
10	1.423	One section of faculty teaching time
3*	1.446	$1,000 of teaching funds
12	1.037	One man year of faculty time

* Resources allocated to Department A as a result of optional quota solution

We will not try to unravel the meaning of all the shadow prices, but will interpret a few.

a. Rows, 4, 5 and 6 as restrictions affect the value of the objective function through whichever research activity would be expanded if these restrictions were relaxed by one unit.

In this particular case, all three restrictions are limiting upon the expansion of Activity A2. A unit of Activity A2 requires one research assistant, one man year of faculty time, and $1,500 of current expense funds and its objective function weight is 3.75 points. As Activity A2 is a linear homogeneous production function, a doubling of the amount of every input will cause a doubling of the output.

We must note that the Row 6 restriction is in $1,000 units, whereas $1,500 is required for each unit of Activity A2. Hence, the Row 6 shadow price of 0.178 per $1,000 is better interpreted as 0.267 per $1,500. We have, then, the following:

		Points:
Row 4:	Shadow price per research assistant:	0.491
Row 5:	Shadow price per faculty man year:	2.991
Row 6:	Shadow price per $1,500 current expense funds:	0.267
	Sum of the three shadow prices:	3.749

Apart from rounding errors, this exhibit makes it perfectly clear that the three restrictions are preventing an expansion in Activity A2, which would increase the objective function by 3.75 points per unit of that activity.

The other shadow prices are not this transparent. Certain relationships, however, suggest the nature of the mechanisms at work:

b. The shadow prices for Row 9 and Row 10 are identical. Their identity implies that the marginal value products of the last unit of faculty time in the undergraduate and graduate teaching programs are equal. The size of the shadow prices, 1.423 points per section of faculty teaching time, suggests that a good part of the effect may come through an increase in M.S. degrees produced by Activity A4, as 0.8762 sections of faculty time in graduate teaching and thesis supervision combined are sufficient to permit an additional M.S. degree, valued at 1.50 points. One section would permit an increase in the objective function of 1.708 points through Activity A4. Offsets probably come through the fact that each M.S. degree awarded by Department A requires three service courses from Department B at an "opportunity cost" of 0.084 points per course (the shadow price corresponding to Row 2). The net result of these two effects would be 1.708 - 3(0.084) = 1.708 - 0.252 = 1.456. Other, more obscure, effects would account for the difference between 1.456 and 1.423.

c. It may be noted that the shadow prices associated with Rows 1 and 8; namely 0.059 and 0.219, are in inverse ratio to the graduate class size of 24 and the thesis "class" size of 6.5. Thus, $\frac{0.059}{0.219} = \frac{6.5}{24} = \frac{1.416}{1.424} = \frac{1}{1}$ except for rounding errors. In Row 1, an additional graduate student-course would require $\frac{1}{24} =$ 0.04167 sections' worth of faculty time. If used to increase M.S. degree output in Activity A4, the objective function would be increased by 0.0712 points (1.708 times 0.04167). The service course demand on Department B would draw this down by 0.252 times 0.04167 or 0.0105 points, for a net effect of about 0.0607 points, very close to the 0.059 of the optimal solution.

We may note also that, in Table 22 of Appendix 3, the shadow prices associated with the teaching budget restriction are the same (1.446) in all three departments. Those associated with graduate instruction in the subject matter of Department A are the same (0.059) in all three departments; and those associated with graduate teaching in the subject matter of Department B are the same (0.084) in two departments, A and B. (Graduate students in Department C do not take courses in Department B.)

The identical shadow prices for a given college-level resource in the two or three departments sharing it indicate that the marginal value products of that resource in its alternative uses have been equalized, a requirement for optimal efficiency as measured by the college-level objective function.

We will not carry the interpretations further. Our main purpose has been to show that fairly complicated programming models can be broken down into smaller components; their mechanisms can be approximately elucidated, and their results monitored at least roughly by direct reasoning. In a real application of the Department A (and college) model, the dean and the department chairmen should each have a good deal of intuitive judgment and experience to aid them in interpreting the results of the computations.

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FORMULATION OF MANAGEMENT SCIENCE MODELS FOR SELECTED PROBLEMS OF COLLEGE ADMINISTRATION

by

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APPENDIXES

Appendix 1

A Model for Allocating a Given Faculty among Alternative Teaching and F   arch Assignments

The model can be defined in terms of (1) a set of constraints which must be satisfied and (2) an objective function the value of which is to be maximized subject to the constraints. We assume that commitments are expressed in terms of units such as "teaching a three credit-hour course for one quarter," and that the amount of time of each faculty member which can be allocated to satisfy the commitments is measured in the same units. Thus, if the ith faculty member were responsible for teaching three three-credit hour courses each quarter, or nine such courses for the academic year as a whole, $a_i = 9$. Similarly, if the jth commitment consists of offering two independent sections of a specified three-hour course in each of three quarters, $b_j = 6$. If the ith faculty member is assigned to teach all six of these offerings of the jth course, we have $x_{ij} = 6$, and the ith faculty member is still available to teach three other courses, presumably one in each quarter.

Suppose we have a department with n faculty members and m different sorts of commitments to fulfill. The constraints faced by this department can be written as:

$$(1) \qquad \sum_{j=1}^{m} x_{ij} \leq a_i , \qquad i = 1, 2, \ldots, n$$

(2) $\sum_{i=1}^{n} x_{ij} = b_j$, $j = 1, 2, \ldots, m$

(3) $x_{ij} \geqq 0$, $i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, m$

The type (1) inequalities insure that the total of the allocations of each faculty member's time does not exceed the amount available for such allocation.

The type (2) equalities insure that all of the commitments are fulfilled.

The type (3) inequalities preclude negative assignments (e.g., assigning some faculty member to teach a negative number of sections of some course). A "negative assignment" makes no sense in this context and would be avoided on pragmatic grounds if the problem were being solved by hand by someone who knew what the numbers signified. If the problem is programmed for a computer, however, the context is lost and the restrictions that $x_{ij} \geqq 0$ for all i and j must be explicitly included in the program.

The x_{ij}'s represent the amount of the ith faculty member's time allocated to the jth commitment. The a_i's indicate the number of units of the ith faculty member's time which is available to the department. The b_j's indicate the number of units of faculty time required by the jth commitment.

If $\sum_{i=1}^{n} a_i < \sum_{j=1}^{m} b_j$ no feasible solution exists, i.e., the department is either overcommitted or understaffed.

If $\sum_{i=1}^{n} a_i \geq \sum_{j=1}^{m} b_j$, feasible solutions exist. If $\sum_{i=1}^{n} a_i > \sum_{j=1}^{m} b_j$, the department is overstaffed or undercommitted. If $\sum_{i=1}^{n} a_i = \sum_{j=1}^{m} b_j$, there is a balance between staff and commitments. In that case the constraints have the same form as the constraints of the so-called "transportation model" and the model can be solved by any of the methods applicable for the solution of such models.[1/]

A. <u>The Objective Function</u>

Nonlinear objective functions may be reasonable in some cases but linear objective functions are usually more desirable for this model. If the objective function is linear and the a_i's and b_j's are all integers the solution vector will, under most methods of solving the model, be linear. A linear objective function could be written as

$$\sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} x_{ij} ,$$

1/ The "transportation model" specifies that there are a_i units of a product at the ith shipping point ($i = 1, 2, \ldots, n$) and that b_j units of the product are needed at the jth destination ($j = 1, 2, \ldots, m$); also

$$\sum_{i=1}^{n} a_i = \sum_{j=1}^{m} b_j .$$

Given the transportation cost, c_{ij}, from each of the n shipping points to each of the m destinations, the problem is to allocate the supply at each shipping point to a destination or destinations such a way that (1) all destination requirements are satisfied and (2) the total transportation cost is minimized; that is,

$$\sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} x_{ij}$$

is a minimum, where x_{ij} is the number of units of the product transported from shipping point i to destination j.

where c_{ij} is a measure of the value per unit resulting when, for example, Course j is taught by Faculty Member i and x_{ij} is the number of units of his time assigned to Course j.

Appendix 2

Models for Computing Optimal Decisions over a Sequence of Years: Recursive Programming and Dynamic Programming

We will outline a recursive programming model in some detail and add briefer comments about a dynamic programming model.

A. Recursive Programming Models

(4) $$\max \sum_{i=1}^{m} c_{it}\, x_{it}$$

subject to

(5) $$(1 - \underline{\beta})x_{i,t-1} \leq x_{i,t} \leq (1 + \bar{\beta})x_{i,t-1}\,, \quad i = 1, 2, \ldots, m$$

(6) $$\sum_{i=1}^{m} a_{ij}x_{it} \leq (1 + \bar{\alpha}) \sum_{i=1}^{m} a_{ij}x_{i,t-1}\,, \quad j = 1, 2, \ldots, k\,.$$

The objective function weights, the c_{it}'s, are usually dependent on past net prices. In many cases they are set equal to last year's net prices. (The "net prices" for year t are equal to year t's output price per unit minus the cost of the resources required to produce one unit of output.)

The type (5) constraints insure that the activity levels chosen in year t do not differ too drastically from the activity levels in year t-1. The $\underline{\beta}_i$'s and $\bar{\beta}_i$'s (both of which are greater than zero and usually less than one) are the "flexibility coefficients." The $\underline{\beta}_i$ and the $\bar{\beta}_i$ associated with any activity i do not have to be equal. For example, we could have

$$(5.1) \qquad (1 - 0.10)x_{i,t-1} \leq x_{i,t} \leq (1 + 0.20)x_{i,t-1}$$

signifying that activity i in year t could not fall more than 10 percent, but could rise as much as 20 percent, from its level in year t-1.

The type (6) constraints limit the increase over last year's levels in the use of certain resources. The $\bar{\alpha}_j$'s are "flexibility coefficients" for these resources. Examples might be various kinds of college building space or legislative appropriations for specified programs.

In the model described by expressions (4), (5) and (6), m equals the number of activities, K equals the number of resource constraints, and a_{ij} indicates the amount of resource j required to produce one unit of output.

The model used in Section III, pages 45-50 of this report could be written as:

$$(7) \qquad \max \sum_{i=1}^{m} c_{it} \, x_{it} - \sum_{i=1}^{m} d_i (x_{it} - x_{i,t-1})^2$$

subject to

$$(8) \qquad \sum_{i=1}^{m} a_{ijl} \, x_{it} \leq b_{j,t+l-1} - \sum_{l=2}^{n} \sum_{i=1}^{m} a_{ijl} \, x_{i,t-l+1}$$

$$j = 1, 2, \ldots, K$$

$$l = 1, 2, \ldots, n$$

(9) $$x_{it} \geqq 0$$

where

(10) $$c_{it} = \sum_{j=1}^{K} a_{ij1} V_{j,t-1} - \left(\frac{\sum\limits_{l=1}^{n} \sum\limits_{j=1}^{K} a_{ijl} V_{j,t-1} - P_i}{\sum\limits_{l=1}^{n} \left| \sum\limits_{j=1}^{K} a_{ijl} V_{j,t-1} \right|} \right)$$

$$\left| \sum_{j=1}^{K} a_{ijl} \, V_{j,t-1} \right|$$

The type (8) constraints are resource constraints which insure that the decisions made in year t do not require more resources than will be available in year t and future years. Ordinarily, the constraints corresponding to future years will not be needed (i.e., the restrictions corresponding to $l = 2, 3, \ldots, n$ will not be needed).

In the present model, m equals the number of activities, K equals the number of resources for which constraints are included, and n equals the number of periods required by the most lengthy production process. Further, a_{ijl} indicates the amount of resource j required during the lth period of the production process in order to produce one unit of output i; $V_{j,t-1}$ is the "shadow price" obtained in the t-lth year for the jth resource; and d_i is the weight associated with deviations in the level of activity i for year t from the level of activity i in year t-l.

The numerical recursive model actually used is as follows:

$$\text{(11)} \quad \max\ c_{1t}x_{1t} + c_{2t}x_{2t} - 0.2(x_{1t}-x_{1t-1})^2 - 0.2(x_{2t}-x_{2t-1})^2$$

subject to

$$\text{(12)} \quad 17.5x_{1,t} - x_{4t} - x_{5t} \leq -17.5x_{1,t-1}$$

$$\text{(13)} \quad x_{4t} + 0.6x_{5t} - 245x_{6t} - 140x_{7t} \leq -17.5(x_{1,t-2}+x_{1,t-3})$$

$$\text{(14)} \quad -17.5x_{3t} + 0.4x_{5t} \leq 0$$

$$\text{(15)} \quad 9x_{2t} - 18x_{6t} - 72x_{7t} \leq -12(x_{2,t-1})$$

$$\text{(16)} \quad -x_{2t} + x_{3t} \leq x_{2,t-1}$$

$$\text{(17)} \quad x_{6t} + x_{7t} \leq 6$$

$$\text{(18)} \quad x_{3t} \leq 2$$

$$\text{(19)} \quad x_{1t} \leq 30$$

$$\text{(20)} \quad x_{2t} \leq 7$$

$$\text{(21)} \quad x_{it} \geq 0 \qquad i = 1, 2, \ldots, 7$$

x_{1t} = number of freshman students admitted in year t.

x_{2t} = number of graduate students admitted in year t.

x_{3t} = number of teaching assistant employed in year t.

x_{4t} = number of undergraduate sections taught solely by faculty members in year t.

x_{5t} = number of undergraduate sections taught partly by graduate students in year t.

x_{6t} = effective number of faculty members assigned to teach seven sections of undergraduate courses and one section of graduate courses per faculty member per year in year t.

x_{7t} = effective number of faculty members assigned to teach four undergraduate sections and four graduate sections per faculty member per year in year t.

Constraint (12) insures that all freshman and sophomores are taught the required number of courses.

Constraint (13) insures that juniors and seniors receive the required amount of instruction.

Constraint (14) insures that the amount of graduate assistant teaching required does not exceed the amount available.

Constraint (15) insures that graduate students receive the required amount of instruction.

Constraint (16) insures that the number of teaching assistants in year t does not exceed the number of graduate students.

Constraint (17) insures that the number of faculty members allocated does not exceed the number available.

Constraint (18) insures that the number of graduate assistants does not exceed the number allowed by the budget.

Constraint (19) insures that the number of freshman admitted does not exceed the number seeking admission.

Constraint (20) insures that the number of new graduate students admitted does not exceed the number seeking admission.

Constraint (21) insures that numbers of persons (x_{1t}, x_{2t}, x_{3t}, x_{6t}, and x_{7t}) and of sections (x_{4t} and x_{5t}) in each category will be non-negative--i.e., positive or zero.

B. Dynamic Programming Model

The model used in computing the admission figures in Table 12, page 52 of the text is as follows:

(22) $$\max \sum_t (c_{1t}x_{1t} + c_{2t}x_{2t})$$

subject to

(23) $$x_{1,t-1} + x_{1t} \leq x_{3t} + x_{4t}$$

(24) $$x_{1,t-3} + x_{1,t-2} + 0.6x_{3t} + x_{4t} + x_{5t} \leq 84$$

(25) $$4x_{2,t-1} + 3x_{2t} - 3x_{5t} \leq 36$$

(26) $$x_{1t} \leq 30$$

(27) $$x_{2t} \leq 7$$

(28) $$x_{1,t-3} \leq 21$$

(29) $$x_{1,t-2} \leq 22$$

(30) $x_{1,t-2} \leq 23$

(31) $x_{2,t-1} \leq 7$

(32) $x_{it} \geq 0, \quad i = 1, 2, 3, 4 \text{ and } 5$.

Constraint (23) insures that all freshmen and sophomore students are taught the required number of courses.

Constraint (24) insures that juniors and seniors receive the required amount of instruction.

Constraint (25) insures that graduate students receive the required amount of instruction.

Table 16 shows the complete model in matrix equation form, with solutions for the activity levels, the shadow prices, and the objective function. Essentially the same information was presented in Table 13 of the test in a more expository form.

The concentration of blocks of elements on the diagonal of the activity matrix with additional elements below the diagonal is characteristic of dynamic programming models.

Table 16. Dynamic Programming Model in Matrix Equation Form, With Solutions

	BS2	BS1	BS0	BS1	MSO	MS1	I1	T1	TR1	BS2	MS2	I2	T2	TR2	BS3	MS3	I3	T3
c_{it}'s:				3.0		2.0				3.0	2.0				3.0	2.0		
J1			1.0	1.0			-1.0	-1.0										
U1	1.0	1.0					.6	1.0	1.0									
G1					4.0	3.0			-3.0									
J2				1.0						1.0		-1.0	-1.0					
U2		1.0	1.0									.6	1.0	1.0				
G2						4.0					3.0			-3.0				
J3										1.0					1.0		-1.0	-1.0
U3			1.0	1.0													.6	1.0
G3											4.0					3.0		
J4															1.0			
U4				1.0						1.0								
G4																4.0		
J5																		
U5										1.0					1.0			
G5																		
J6																		
U6															1.0			
G6																		
J7																		
U7																		
G7																		
J8																		
U8																		
G8																		
J9																		
U9																		
G9																		
J10																		
U10																		
G10																		
LB/	21.0	22.0	23.0		7	1					1					1		
UB/				30		7	50		48	30	7	50		48	30	7	50	
Activity levels x_{it}'s:	21	22	23	30		7	50	3	4	24	7	50	5	4	23	5	48	0

	TR3	BS4	MS4	I4	T4	TR4	BS5	MS5	I5	T5	TR5	BS6	MS6	I6	T6	TR6	BS7
c_{it}'s:		3.0	2.0				3.0	2.0				3.0	2.0				3.0
J1																	
U1																	
G1																	
J2																	
U2																	
G2																	
J3																	
U3	1.0																
G3	-3.0																
J4		1.0		-1.0	-1.0												
U4				.6	1.0	1.0											
G4			3.0			-3.0											
J5		1.0					1.0		-1.0	-1.0							
U5									.6	1.0	1.0						
G5			4.0					3.0			-3.0						
J6							1.0					1.0		-1.0	-1.0		
U6		1.0												.6	1.0	1.0	
G6								4.0					3.0			-3.0	
J7												1.0					1.0
U7		1.0					1.0										
G7													4.0				
J8																	1.0
U8							1.0					1.0					
G8																	
J9																	
U9												1.0					1.0
G9																	
J10																	
U10																	1.0
G10																	
LB/			1					1					1				
UB/	48	30	7	50		48	30	7				30	7	50		48	30
Activity level																	
x_{it}'s:	2	23	6	47	0	1	24	3	12	36	1	25	7	50	0	0	26

	MS7	I7	T7	TR7	BS8	MS8	I8	T8	TR8	BS9	MS9	I9	T9	TR9	BS10	MS10	I10
c_{it}'s	2.0				3.0	2.0				3.0	2.0						
J1																	
U1																	
G1																	
J2																	
U2																	
G2																	
J3																	
U3																	
G3																	
J4																	
U4																	
G4																	
J5																	
U5																	
G5																	
J6																	
U6																	
G6																	
J7		-1.0	-1.0														
U7		.6	1.0	1.0													
G7	3.0			-3.0													
J8					1.0		-1.0	-1.0									
U8							.6	1.0	1.0								
G8	4.0					3.0			-3.0								
J9					1.0					1.0		-1.0	-1.0				
U9												.6	1.0	1.0			
G9						4.0					3.0			-3.0			
J10										1.0					1.0		-1.0
U10					1.0												.6
G10											4.0					3.0	
LB/	1				24	1				25	1				26		
UB/	7	50		48		7	50		48		7	50		48		7	50
Activity levels x_{it}'s	7	50	2	4	24	6	50	0	4	25	6	49	0	3	26	7	50

c_{it}'s:	T10	TR10		b_i		U (Shadow Prices)
J1			≤	0	J1	0
U1			≤	84	U1	0
G1			≤	36	G1	0
J2			≤	0	J2	0.85
U2			≤	84	U2	0.85
G2			≤	36	G2	0.28
J3			≤	0	J3	0.40
U3			≤	84	U3	0.67
G3			≤	36	G3	0.22
J4			≤	0	J4	0.60
U4			≤	84	U4	1.00
G4			≤	36	G4	0.33
J5			≤	0	J5	0.75
U5			≤	84	U5	0.75
G5			≤	36	G5	0.25
J6			≤	0	J6	0.85
U6			≤	84	U6	1.25
G6			≤	36	G6	0.31
J7			≤	0	J7	0.40
U7			≤	84	U7	0.40
G7			≤	36	G7	0.13
J8			≤	0	J8	0.91
U8			≤	84	U8	1.00
G8			≤	36	G8	0.33
J9			≤	0	J9	0.45
U9			≤	84	U9	0.75
G9			≤	36	G9	0.25
J10	-1.0		≤	0	J10	0.94
U10	1.0	1.0	≤	84	U10	0.94
G10		-3.0	≤	36	G10	0.31
LB/						
UB/		48				
Activity levels x_{it}'s	1	3				

Appendix 3

A Two-Level Decision Model: Interaction between Dean and Department Chairmen in Planning Resource Allocation

Under a two-level decision-making scheme each department may be faced with a problem of the form (for the ith department):

$$(33) \qquad \max \sum_{j=1}^{m} c_{ij}\, x_{ij}$$

subject to

$$(34) \qquad \sum_{j=1}^{m_i} a_{ijK} x_{ij} \leq b_{iK}\,, \qquad K = 1, 2, \ldots, s$$

$$(35) \qquad \sum_{j=1}^{m_i} d_{ijl} x_{ij} \leq u_{il}\,, \qquad l = 1, 2, \ldots, t$$

$$(36) \qquad x_{ij} \geq 0\,. \qquad j = 1, 2, \ldots, m_i$$

The c_{ij}'s are objective function weights, the x_{ij}'s are the activity levels for activities of the ith department, the a_{ijK}'s and d_{ijl}'s are technical coefficients which indicate the amounts of resources K and l used when the jth activity (of the ith department) is operated at the unit level. Further, t equals the number of different resources allocated by the college dean, s_i equals the number of constraints faced by department i with respect to resources which are specialized to it and are not useable by other departments, and m_i equals the number of activities available to department i.

Constraints (34) are constraints on resources (or outputs) used (or produced) only by the ith department.

Constraints (35) are constraints on resources (or outputs) which could be used (or produced) by other departments and which are allocated by the college dean.

Constraints (36) reflect the fact that negative activity levels are not permitted.

The problem faced by the college dean has the form:

$$(37) \qquad \max \sum_{i=1}^{n} \sum_{j=1}^{m_i} c_{ij} \, x_{ij}$$

subject to

$$(38) \qquad \sum_{i=1}^{n} u_{il} \leq b_l \, , \qquad l = 1, 2, \ldots, t$$

and subject to restrictions (34), (35) and (36) being satisfied for all departments.

Here, n equals the number of departments in the college, u_{il} indicates the amount of the lth resource which is allocated to the ith department, and b_l indicates the amount of the lth resource which the college dean has available for allocation.

The constraints (38) insure that the totals of the allocations made by the college dean do not exceed the total amounts available.

The decision process that could be used to obtain an optimum set of quotas consists of several phases each of which can be described by describing the Nth phase. During the Nth phase the college dean solves a problem of the form:[1/]

$$(39) \qquad Z^N = \min \left[\sum_{l=1}^{t} b_l V_l + \sum_{i=1}^{n} w_i \right]$$

subject to

$$(40) \qquad \sum_{l=1}^{t} u_{il}^K V_l + w_i \geq r_i^K \qquad i = 1, 2, \ldots, n$$
$$K = 0, 1, 2, \ldots, N-1$$

$$(41) \qquad V_l \geq 0 . \qquad l = 1, 2, \ldots, t$$

The solution values are designated as V_l^N, $l = 1, 2, \ldots, t$, and w_i^N, $i = 1, 2, \ldots, n$. The V_l^N's are the college dean's current estimates of the shadow prices (marginal values) of the resources which he allocates. The w_i^N's are his current estimates of those resources (and output requirements) which can only be used (or produced) by the individual departments.

He reports the V_l^N's and w_i^N's to the departments. They solve problems which (for the ith department) have the form:

$$(42) \qquad Z_i^N = \max \left[\sum_{j=1}^{m_i} c_{ij} x_{ij} - \sum_{l=1}^{t} u_{il} V_l^N \right] - w_i^N$$

subject to constraints (34), (35) and (36). If Z_i^N is greater than zero the x_{ij}^N's and u_{il}^N's are set equal to the solution values of the

1/ The dual of this problem is ordinarily easier to solve and can be used to obtain the solution to (39) through (41).

x_{ij}'s and u_{il}'s. They also then compute r_i^N by setting it equal to $\sum_{j=1}^{m_i} c_{ij}x_{ij}^N$ and report the values of the u_{il}^N's and of r_i^N to the college dean. If Z_i^N is equal to zero the ith department reports only this fact to the college dean.

In order to initiate this process it is necessary to know initially (at the beginning of phase 1) some feasible values for the u_{il}^o's and r_i^o's. Ordinarily this information would be known by the college dean or could be obtained by modifying plans (allocations) for previous periods. If these values cannot be supplied by the college dean the decision-making process can be modified for as many phases as are required to obtain a feasible solution. The details of this modification can be found in McCamley (15 , pp. 106-107). Essentially the modification amounts, for the college dean's part of the Nth phase, to

$$\text{minimizing } \left[\sum_{l=1}^{t} b_l V_l + \sum_{i=1}^{n} w_i\right]$$

subject to

$$\sum_{l=1}^{t} u_{il}^K V_l + w_i \geq 0 \ , \qquad \begin{array}{l} i = 1, 2, \ldots, n \\ K = 1, 2, \ldots, N-1 \end{array}$$

$$0 \leq V_l \leq 1 \ , \qquad l = 1, 2, \ldots, t$$

$$w_i \geq -1 \ . \qquad i = 1, 2, \ldots, n$$

The departments react as before except that $Z_i^N = \max\left[-\sum_{l=1}^{t} u_{il}\right] - w_i^N$ subject to constraints (34), (35) and (36).

The decision process continues until at the end of some phase, say the Mth, Z_i^M is equal to zero for all relevant i. The college dean would then solve a problem of the form:

$$(43) \qquad Z^M = \max \left[\sum_{i=1}^{n} \sum_{K=1}^{M-1} r_i^K \; \lambda_i^K \right]$$

subject to

$$(44) \qquad \sum_{i=1}^{n} \sum_{K=1}^{M-1} u_{i1}^K \lambda_i^K \leq b_1 \qquad\qquad 1 = 1, 2, \ldots, t$$

$$(45) \qquad \sum_{K=1}^{M-1} \lambda_i^K = 1 \, , \qquad\qquad i = 1, 2, \ldots, n$$

$$(46) \qquad \lambda_i^K \geq 0 \, , \qquad\qquad i = 1, 2, \ldots n, \quad K = 1, 2, \ldots, M-1$$

(This, of course, the dual of the Mth phase version of the problem defined by (39) through (41) and therefore the solution would already be known to him.) Designate by $\hat{\lambda}_i^K$'s the values of the λ_i^K's which solve (43) through (46). An optimum set of quotas could then be obtained by setting

$$u_{i1} = \sum_{K=1}^{M-1} u_{i1}^K \lambda_i^K \qquad\qquad \begin{array}{l} \text{for } i = 1, 2, \ldots, n \\ \text{and } 1 = 1, 2, \ldots, t \end{array}$$

The procedure described above must be modified if unbounded solutions ($Z_i^K \rightarrow \infty$) are obtained by any of the departments at any stage. This modification is described in Dantzig (7 , pp. 453-454).

A. Tabular Presentations of the Model and Its Solution

Tables 17, 18 and 19 present the models for Departments A, B and C respectively. The first two or three rows in each model represent the college-level resources for which optimal interdepartmental allocations are to be determined.

In the Department A model, the x_{Aj}'s have the following meanings:

x_{A1}, x_{A2} and x_{A3} are the activity levels which correspond to the research activities.

x_{A4} = the number of M.S. degrees per year awarded to students providing their own support.

x_{A5} = the number of M.S. degrees per year awarded to students who obtain financial support by working as research assistants.

x_{A6} = the number of M.S. degrees per year awarded to students who obtain financial support by working as teaching assistants.

x_{A7} = the number of undergraduate sections per year taught solely by faculty members.

x_{A8} = the number of undergraduate sections per year taught jointly by faculty members and graduate teaching assistants.

x_{A9} = the number of graduate sections taught per year.

Table 17. Department A Model

Row No.		Research Activities			M.S. Degree Activities			Teaching Activities				Faculty Allocation Activities				Restrictions b vector
	Activity Levels:	x_{A1}	x_{A2}	x_{A3}	x_{A4}	x_{A5}	x_{A6}	x_{A7}	x_{A8}	x_{A9}	x_{A10}	x_{A11}	x_{A12}	x_{A13}		
	Objective function Weights (c_{Aj}'s):	5.0	3.75	2.75	1.5	1.5	1.5									
		A1	A2	A3	A4	A5	A6	A7	A8	A9	A10	A11	A12	A13		
1					10	10	10			-24					≤	u_{A1}
2					3	3	3								≤	u_{A2}
3							5600					7400	7400		≤	u_{A3}
4		3	1			-2									≤	0
5		1	1	1								-1/3	-1/3	-1	≤	0
6		3000	1500	750		5500						3600	3600	11,000	≤	40,000
7								-35	-30						≤	1850
8					3	3	3				-6.5				≤	0
9								1	.5			-6	-4		≤	0
10										1	1	-2	-4		≤	0
11							-10		.5						≤	0
12												1	1	1	≤	5.5
13					1										≤	2.0

Table 18. Department B Model

Row No.		Research Activities		M.S. Degree Activities		Teaching Activities				Faculty Allocation Activities				Restrictions
	Activity Levels:	x_{B1}	x_{B2}	x_{B3}	x_{B4}	x_{B5}	x_{B6}	x_{B7}	x_{B8}	x_{B9}	x_{B10}	x_{B11}		b vector
	Objective function weights (c_{Bj}'s):	2.1	2.4	1.75	1.75									
		B1	B2	B3	B4	B5	B6	B7	B8	B9	B10	B11		
1				3	3								$\leq$	u_{B1}
2				10	10			-15.0					$\leq$	u_{B2}
3					5200					8500	8500		$\leq$	u_{B3}
4		1	1							-.15	-.15	-1	$\leq$	0
5		1000	2600							1500	1500	10,000	$\leq$	20,000
6						-30	-25						$\leq$	2,750
7				3	3				-7				$\leq$	0
8						1	.6			-8.5	-6.0		$\leq$	0
9								1	1	-1.5	-4.0		$\leq$	0
10					-10		.4						$\leq$	0
11										1	1	1	$\leq$	8.5
12				1									$\leq$	1

Table 19. Department C Model

A C T I V I T I E S

Row No.	Research Activities			M.S. Degree Activities			Teaching Activities				Faculty Allocation Activities				Restrictions b vector
Activity Levels:	x_{C1}	x_{C2}	x_{C3}	x_{C4}	x_{C5}	x_{C6}	x_{C7}	x_{C8}	x_{C9}	x_{C10}	x_{C11}	x_{C12}	x_{C13}		
Objective Function Weights: (c_{Cj}'s):	5.4	4.75	3.0	2.0	2.0	2.0									
	C1	C2	C3	C4	C5	C6	C7	C8	C9	C10	C11	C12	C13		
1				4	4	4								≤	U_{C1}
2						5500					9000	9000		≤	U_{C3}
3	3	2		-2										≤	0
4	1	1		1							-.25	-.25	-1.0	≤	0
5	3000	1750	500		5400						3000	3000	12,000	≤	45,000
6							-40	-35						≤	-2,250
7				9.0	9	9			-18					≤	- 210
8				3	3	3				-7.5				≤	0
9							1	.7			-7	-5		≤	0
10									1	1	-2	-4		≤	0
11						-12		.3						≤	0
12											1	1	1	≤	7.5
13				1										≤	3.0

x_{A10} = the number of units (section equivalents) of faculty time devoted to thesis supervision.

x_{A11}, x_{A12} and x_{A13} are the activity levels for the various types of faculty time allocations among research, graduate teaching and undergraduate teaching.

In the Department B model, the x_{Bj}'s have the following meanings:

x_{B1} and x_{B2} are the activity levels for the research activities.

x_{B3} = the number of M.S. degrees per year awarded to students providing their own support.

x_{B4} = the number of M.S. degrees per year awarded to students who are also teaching assistants.

x_{B5}, x_{B6} and x_{B7} are the numbers of each of the various types of sections taught per year.

x_{B8} = the number of units (section equivalents) of faculty time devoted to thesis supervision.

x_{B9}, x_{B10} and x_{B11} are the activity levels for the faculty time allocation activities (allocations among research, graduate teaching and undergraduate teaching).

In the Department C model, the x_{cj}'s have the following meanings:

x_{c1}, x_{c2} and x_{c3} are the activity levels for the research activities.

x_{c4} = the number of M.S. degrees per year awarded to students providing their own support.

x_{c5} = the number of M.S. degrees per year awarded to students who are also research assistants.

x_{c6} = the number of M.S. degrees per year awarded to students who are also teaching assistants.

x_{c7}, x_{c8} and x_{c9} are the numbers of each of the various types of sections taught.

x_{c10} = the number of units (section equivalents) of faculty time per year devoted to thesis supervision.

$x_{c}11$, $x_{c}12$ and x_{c13} are the activity levels for the faculty time allocation activities (allocations among research, graduate teaching and undergraduate teaching).

Table 20 combines the models for all three departments. The top eight rows involve the three resources for which optimal allocations among the three departments are to be computed by the two-level decision process. The college dean faces one major restriction, U3, a total teaching budget of $220,000 assigned to him by the university president. The other two restrictions, U1 and U2, are in a sense self-imposed by the college dean, who uses their suballocations among the departments as tools for balancing up or equalizing the marginal value products of the graduate teaching programs in the three departments.

Table 21 displays the values of key magnitudes (objective functions, shadow prices and tentative quotas) at successive phases of the

Table 10. Complete Model of Departments A, B and C with Department-Level and College-Level Restrictions

	Row Number	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10	A11	A12	A13
		Department A Activities: ← (c_{Aj}'s)												
		5.00	3.75	2.75	1.50	1.50	1.50							
College-Level Restrictions	A1(U1)				10	10	10			-24				
	B1(U1)													
	C1(U1)													
	A2(U2)				3	3	3							
	B2(U2)													
	A3(U3)						5600					7400	7400	
	B3(U3)													
	C3(U3)													
Department A Restrictions	4a	3	1			-2								
	5a	1	1	1								-1/3	-1/3	-1
	6a	3000	1500	750		5500						3600	3600	11,000
	7a							-35	-30					
	8a				3	3	3				-6.5			
	9a							1	0.5			-6	-4	
	10a									1	1	-2	-4	
	11a						-10		0.5					
	12a											1	1	1
	13a				1									
Department B Restrictions	4b													
	5b													
	6b													
	7b													
	8b													
	9b													
	10b													
	11b													
	12b													
Department C Restrictions	3c													
	4c													
	5c													
	6c													
	7c													
	8c													
	9c													
	10c													
	11c													
	12c													
	13c													

Table 20. (Cont.)

	Row Number	2.10 B1	2.40 B2	1.75 B3	1.75 B4	Department B Activities: B5	B6	B7	B8	(c_{Bj}'s) B9	B10	B11
College-Level Restrictions	A1(U1)											
	B1(U1)			3	3							
	C1(U1)											
	A2(U2)											
	B2(U2)			10	10			-15				
	A3(U3)											
	B3(U3)				5200					8500	8500	
	C3(U3)											
Department A Restrictions	4a											
	5a											
	6a											
	7a											
	8a											
	9a											
	10a											
	11a											
	12a											
	13a											
Department B Restrictions	4b	1	1							-0.15	-0.15	-1
	5b	1000	2600							1500	1500	10,000
	6b					-30	-25					
	7b			3	3				-7			
	8b					1	0.6			-8.5	-6.0	
	9b							1	1	-1.5	-4.0	
	10b				-10		0.4					
	11b									1	1	1
	12b			1								
Department C Restrictions	3c											
	4c											
	5c											
	6c											
	7c											
	8c											
	9c											
	10c											
	11c											
	12c											
	13c											

Table 20. (Cont.)

	Row Number	5.40 C1	4.75 C2	3.00 C3	2.00 C4	2.00 C5	2.00 C6	C7	C8	C9	C10	C11	C12	C13
		Department C Activities: ← (c_{Cj}'s)												
College-Level Restrictions	A1(U1)													
	B1(U1)													
	C1(U1)				4	4	4							
	A2(U2)													
	B2(U2)													
	A3(U3)													
	B3(U3)													
	C3(U3)						5500					9000	9000	
Department A Restrictions	4a													
	5a													
	6a													
	7a													
	8a													
	9a													
	10a													
	11a													
	12a													
	13a													
Department B Restrictions	4b													
	5b													
	6b													
	7b													
	8b													
	9b													
	10b													
	11b													
	12b													
Department C Restrictions	3c	3	2			-2								
	4c	1	1	1								-0.25	-0.25	-1
	5c	3000	1750	500		5400						3000	3000	12,000
	6c							-40	-35					
	7c				9	9	9			-18				
	8c				3	3	3				-7.5			
	9c							1	0.7			-7	-5	
	10c									1	1	-2	-4	
	11c						-12		0.3					
	12c											1	1	1
	13c				1									

Table 20. (Cont.)

	Row Number		Department Restrictions b Vector		College Restrictions
College-Level Restrictions	A1(U1)	≤	UA1		
	B1(U1)	≤	UB1	=	U1
	C1(U1)	≤	UC1		
	A2(U2)	≤	UA2	=	U2
	B2(U2)	≤	UB2		
	A3(U3)	≤	UA3		
	B3(U3)	≤	UB3	=	U3 ≤ $220,000
	C3(U3)	≤	UC3		
Department A Restrictions	4a	≤	0		
	5a	≤	0		
	6a	≤	40,000		
	7a	≤	-1,850		
	8a	≤	0		
	9a	≤	0		
	10a	≤	0		
	11a	≤	0		
	12a	≤	5.5		
	13a	≤	2.0		
Department B Restrictions	4b	≤	0		
	5b	≤	20,000		
	6b	≤	-2,750		
	7b	≤	0		
	8b	≤	0		
	9b	≤	0		
	10b	≤	0		
	11b	≤	8.5		
	12b	≤	1.0		
Department C Restrictions	3c	≤	0		
	4c	≤	0		
	5c	≤	45,000		
	6c	≤	-2,250		
	7c	≤	-210		
	8c	≤	0		
	9c	≤	0		
	10c	≤	0		
	11c	≤	0		
	12c	≤	7.5		
	13c	≤	3.0		

Table 21. Two-Level Decision Model: Values of Key Magnitudes (Objective Functions, Shadow Prices and Tentative Quotas) at Successive Phases of the Communication Process between the College Dean and Departments A, B and C

	Phase Number (K)							
Variable	0	1	2	3	4	5	6	7
				COLLEGE DEAN				
V_1^K		0	0	0.10	.04	.06	.05	.06
V_2^K		0	0	0.05	.05	.09	.08	.08
$1000V_3^K$		0	1.12	0.25	.78	1.53	1.42	1.45
W_A^K		12.70	-50.75	13.33	-20.28	-61.26	-55.10	-56.03
W_B^K		10.48	-92.59	-6.11	-52.73	-116.55	-106.64	-108.96
W_C^K		1.07	-77.85	4.30	-32.41	-86.77	-78.33	-80.48
$Z:^K$lower est.[a]		24.25	25.28	53.37	57.82	58.21	58.29	58.38
$Z:^K$upper est.[b]		166.72	63.65	63.65	59.07	58.60	58.51	58.38
				DEPARTMENT A				
T_A^K	12.70	51.88	20.77	17.75	20.75	17.38	20.38	
U_{A1}^K	-166.62	285.27	-82.21	-194.34	-152.18	-133.59	-91.44	
U_{A2}^K	16.33	85.58	23.41	17.32	23.32	16.10	22.10	
U_{A3}^K	56.63	174.20	55.29	57.97	57.97	55.29	55.29	
Z_A^K		39.18	9.57	7.29	0.88	0.27	0.22	0
				DEPARTMENT B				
r_B^K	10.48	80.64	13.27	12.51	13.26	12.32		
U_{B1}^K	14.7	133.0	16.2	16.2	16.2	14.59		
U_{B2}^K	-149.36	443.32	-86.00	-196.29	-141.18	-103.08		
U_{B3}^K	92.0	297.58	88.88	95.13	92.01	89.22		
Z_B^K		70.16	6.29	3.64	0.37	0.12	0	0

Table 21 (continued)

Variable	Phase Number (K) 0	1	2	3	4	5	6	7
			DEPARTMENT C					
r_C^K	1.07	34.20	25.21	25.58				
U_{C1}^K	2.14	48.15	28.43	30.12				
U_{C2}^K	0	0	0	0				
U_{C3}^K	70.45	100.02	71.90	72.28				
Z_C^K		33.13	22.51	0.12	0	0	0	0

a/ The lowest estimate is equal to Z^K for Phase K.

b/ The upper estimate for Phase K is given by

$$\min_{1 \leq h \leq K} \left[Z^K + Z_A^K + Z_B^K + Z_C^K \right]$$

communication process between the dean and the department chairmen. Finally, Table 22 displays the activity levels (x_{ij}'s) and shadow prices (v_{ij}'s) resulting from the optimal solution (end of Phase 7) of the overall college model.

In Table 6, it should be noted that the activity levels (x_{Aj}, x_{Bj} and x_{Cj}'s) are associated with the <u>columns</u> of the corresponding department models and the shadow prices (v_{Aj}, v_{Bj} and v_{Cj}'s) are associated with the <u>rows</u>. There are 37 columns (activities) in the three departments and there are 38 rows (shadow prices). The near-equality of the number of rows and the number of columns is a more or less coincidental byproduct of our attempt to keep the model small. We could, for example, have included a very large number of alternative class sizes in each department model, with each class size constituting a separate activity. This would have led to perhaps 100 possible activities (columns) but the number of restrictions (rows) might have remained at 38. However, not more than 38 activities would have appeared at nonzero levels in the optimal solution.

Table 22. Two-Level Decision Models: Activity Levels (x_{ij}'s) and Shadow Prices (v_{ij}'s) in Optimal Solution (at End of Phase 7).

Activity Variable Identification	Activity Levels in Solution for College Model	Shadow Price Variable Identification	Shadow Prices (Changes in Objective Function Value per Unit Change in the Corresponding Row Restriction)	
			Solution Values in Department Model	Solution Values in Overall College Model 1/
		DEPARTMENT A		
x_{A1}	1.47	V_{A1}	0.004	0.059
x_{A2}	0.50	V_{A2}	0.475	0.084
x_{A3}	0.0	$1000V_{A3}$	0.0	1.446
x_{A4}	1.66	V_{A4}	0.491	0.491
x_{A5}	2.45	V_{A5}	2.991	2.991
x_{A6}	3.08	V_{A6}	0.179	0.178
x_{A7}	0.0	V_{A7}	0.001	0.037
x_{A8}	61.67	V_{A8}	0.013	0.219
x_{A9}	8.18	V_{A9}	0.084	1.423
x_{A10}	3.32	V_{A10}	0.084	1.423
x_{A11}	4.83	V_{A11}	0.0	0.810
x_{A12}	0.46	V_{A12}	1.027	1.037
x_{A13}	0.21	V_{A13}	0.0	0.0

Table 22 (continued)

		DEPARTMENT B		
x_{B1}	0.0	V_{B1}	0.0	0.059
x_{B2}	1.59	V_{B2}	0.107	0.084
x_{B3}	1.0	$1000V_{B3}$	1.792	1.446
x_{B4}	4.34	V_{B4}	2.085	1.968
x_{B5}	1.35	V_{B5}	0.121	0.166
x_{B6}	108.38	V_{B6}	0.053	0.042
x_{B7}	12.66	V_{B7}	0.228	0.179
x_{B8}	2.29	V_{B8}	1.598	1.256
x_{B9}	7.03	V_{B9}	1.598	1.256
x_{B10}	1.10	V_{B10}	0.932	0.732
x_{B11}	0.37	V_{B11}	0.874	0.307
		V_{B12}	0.0	0.197
		DEPARTMENT C		
x_{C1}	0.51	V_{C1}	0.0	0.059
x_{C2}	1.74	$1000V_{C2}$	2.089	1.446
x_{C3}	0.0	V_{C3}	0.444	0.384
x_{C4}	3.00	V_{C4}	3.573	3.610
x_{C5}	2.50	V_{C5}	0.165	0.213
x_{C6}	1.61	V_{C6}	0.053	0.036
x_{C7}	0.0	V_{C7}	0.124	0.085
x_{C8}	64.29	V_{C8}	0.296	0.205
x_{C9}	15.22	V_{C9}	2.222	1.534
x_{C10}	2.84	V_{C10}	2.222	1.534
x_{C11}	4.98	V_{C11}	0.958	0.631
x_{C12}	2.02	V_{C12}	1.599	1.055
x_{C13}	0.49	V_{C13}	0.0	0.382

1/ These values are associated with the x_{ij}'s in the first column, and are the ones used in the text discussion, pages .